

QUANTUM EXTREMAL LOOP WEIGHT MODULES AND MONOMIAL CRYSTALS

MATHIEU MANSUY

ABSTRACT. In this paper we construct a new family of representations for the quantum toroidal algebras of type A_n . The definition of extremal loop weight modules of quantum toroidal algebras was proposed in [23]. We construct extremal loop weight modules associated to analogues of level 0 fundamental weights ϖ_ℓ for $\mathcal{U}_q(sl_{n+1}^{tor})$ when $n = 2r + 1$ is odd and $\ell = 1$ or $\ell = r + 1$, called level 0 extremal fundamental loop weight modules. To do it, we relate monomial realizations of level 0 extremal fundamental weight crystals with integrable representations of $\mathcal{U}_q(sl_{n+1}^{tor})$. The construction is based on the combinatorial study of these crystals: we introduce promotion operators for the level 0 extremal fundamental weight crystals, corresponding to the cyclic symmetry of the Dynkin diagram of type $A_n^{(1)}$. They are used to define an action of the quantum toroidal algebras at the level of representations. By specializing q at roots of unity ϵ , we get finite-dimensional modules of $\mathcal{U}_\epsilon(sl_{n+1}^{tor})$. In general, we give a conjectural process to construct extremal loop weight modules from monomial realizations of crystals.

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1. INTRODUCTION

Let us consider a finite-dimensional simple Lie algebra \mathfrak{g} and its associated quantum affine algebra $\mathcal{U}_q(\hat{\mathfrak{g}})$. A famous result of Beck and Drinfeld [3, 11] says that $\mathcal{U}_q(\hat{\mathfrak{g}})$ has two realizations: first as the quantized enveloping algebra of the affine Lie algebra $\hat{\mathfrak{g}}$ and second as the Drinfeld quantum affinization of the quantum group $\mathcal{U}_q(\mathfrak{g})$.

The representation theory of the quantum affine algebras have been intensively studied (see, among others, [1, 5, 7, 9, 17, 19, 35, 39]). Kashiwara [31] has defined a class

of integrable representations $V(\lambda)$ of these algebras, called extremal weight modules, parametrized by an integrable weight λ with crystal basis $\mathcal{B}(\lambda)$. When λ is dominant, we obtain the simple integrable module of highest weight λ . But in general $V(\lambda)$ is not simple and it is neither of highest weight nor of lowest weight. These representations were the subject of numerous papers (see [4, 5, 27, 31, 33, 37, 38, 41]) and have a particular importance because they have finite-dimensional quotients for some weight λ . Kashiwara has proved in this way the existence of crystal bases for the finite-dimensional fundamental representations of $\mathcal{U}_q(\hat{\mathfrak{g}})$ (for a special choice of the spectral parameter).

The quantum affine algebra $\mathcal{U}_q(\hat{\mathfrak{g}})$ is also a quantum Kac-Moody algebra and thus can be affinized again by the Drinfeld quantum affinization process. One gets a toroidal (or double affine) quantum algebra $\mathcal{U}_q(\mathfrak{g}^{tor})$ which is not a quantum Kac-Moody algebra anymore and can not be affinized again by this process (it can be viewed as “the terminal object” in this construction). These algebras were first introduced by Ginzburg-Kapranov-Vasserot in type A [20] and then in the general context [28, 39]. In type A, they are in Schur-Weyl duality with elliptic Cherednik algebras [46].

The representation theory of these algebras is very interesting and has been intensively studied (see for example [12, 13, 14, 15, 21, 23, 25, 36, 47] and references therein). In the spirit of works of Kashiwara, Hernandez [23] proposed the definition of extremal loop weight modules for $\mathcal{U}_q(\mathfrak{g}^{tor})$. The main motivation is to construct finite-dimensional representations of the quantum toroidal algebra at roots of unity. He constructs the first example of such a module for $\mathcal{U}_q(sl_4^{tor})$ which is neither of ℓ -highest weight nor of ℓ -lowest weight. This module is generated by a ℓ -weight vector of ℓ -weight an analogue of the level 0 fundamental weight $\varpi_1 = \Lambda_1 - \Lambda_0$ in the toroidal case. By specializing q at roots of unity ϵ , he obtains finite-dimensional representations of $\mathcal{U}_\epsilon(sl_4^{tor})$.

In the present paper, we construct a new family of extremal loop weight modules for the quantum toroidal algebras of type A_n : we define extremal loop weight modules (called level 0 extremal fundamental loop weight modules) generated by a vector of ℓ -weight an analogue of the level 0 fundamental weight $\varpi_\ell = \Lambda_\ell - \Lambda_0$ for $\mathcal{U}_q(sl_{n+1}^{tor})$ when $n = 2r + 1$ is odd and $\ell = 1$ or $\ell = r + 1$ (Theorem 4.1). This construction is based on the monomial realizations of level 0 extremal fundamental weight crystals $\mathcal{B}(\varpi_\ell)$. We relate these monomial crystals with integrable representations of $\mathcal{U}_q(sl_{n+1}^{tor})$. For that we study the combinatorics of these crystals: we introduce promotion operators for $\mathcal{B}(\varpi_\ell)$ ($1 \leq \ell \leq n$) and we explicit them in terms of monomials. These operators play an important role in our work: on the one hand, at the level of crystals, they are used to check that these monomial crystals are closed when $\ell = 1$ or $\ell = r + 1$ (see Definition 3.6 for this notion, related to the theory of q -characters), and on the other hand, at the level of representations, they enable us to define the action of the quantum toroidal algebra. We show that the representations we constructed are irreducible and, as modules over the horizontal subalgebra, they are isomorphic to the fundamental extremal weight modules $V(\varpi_\ell)$. We give explicit formulas for the action, in terms of constants of the associated monomial crystal. By specializing the quantum parameter q at roots of unity ϵ , we get new irreducible finite-dimensional representations of $\mathcal{U}_\epsilon(sl_{n+1}^{tor})$. When ℓ is different of 1 or $r + 1$, the corresponding monomial crystals are

not closed and it is not possible to make the same construction. We give a conjectural process to define other extremal loop weight modules in this situation: as an example, we construct an extremal loop weight module of $\mathcal{U}_q(sl_4^{tor})$ generated by a vector of ℓ -weight an analogue of $2\varpi_1$.

Let us describe the methods used in this paper in more detail. In [34, 40], Kashiwara and Nakajima have defined a crystal \mathcal{M} (called monomial crystal) where the vertices are monomials of $\mathbb{Z}[Y_{i,l}^{\pm}]_{i \in I, l \in \mathbb{Z}}$. They determined monomial realizations $\mathcal{M}(m)$ (subcrystal of \mathcal{M} generated by the monomial m) of crystals of finite type: when $m = Y_{i,l}$ ($1 \leq i \leq n$), $\mathcal{M}(m)$ is isomorphic to the finite $\mathcal{U}_q(sl_{n+1})$ -crystal $\mathcal{B}(\Lambda_i)$. These results were extended in [27] to the monomial realizations of extremal weight crystals of $\mathcal{U}_q(\hat{sl}_{n+1})$ for $n = 2r + 1$ odd: if $m = Y_{\ell,0}Y_{0,\ell}^{-1}$ ($1 \leq \ell \leq n$), the monomial crystal $\mathcal{M}(m)$ is isomorphic to the level 0 extremal fundamental weight crystal $\mathcal{B}(\varpi_\ell)$.

The monomial crystals are closely related to the theory of q -characters of the finite-dimensional representations of $\mathcal{U}_q(\hat{sl}_{n+1})$. More precisely, the set of monomials occurring in the $\mathcal{U}_q(sl_{n+1})$ -crystal $\mathcal{M}(Y_{i,l})$ and the set of ℓ -weights of the fundamental $\mathcal{U}_q(\hat{sl}_{n+1})$ -module $V(Y_{i,l})$ of ℓ -highest weight $Y_{i,l}$ are equal (we use here the identification of Frenkel-Reshetikhin [19] between ℓ -weights and monomials in $\mathbb{Z}[Y_{i,l}^{\pm}]_{i \in I, l \in \mathbb{Z}}$). Motivated by these facts, Hernandez [23] constructed the first example of extremal loop weight modules for $\mathcal{U}_q(sl_4^{tor})$. He used the monomial crystal $\mathcal{M}(Y_{1,0}Y_{0,1}^{-1})$ to construct a representation whose q -character is the sum of monomials occurring in $\mathcal{M}(Y_{1,0}Y_{0,1}^{-1})$. We use the same technical feature in this paper. We propose to relate the monomial crystals $\mathcal{M}(Y_{\ell,0}Y_{0,\ell}^{-1})$ of $\mathcal{U}_q(\hat{sl}_{n+1})$ (where $n = 2r + 1$ is supposed to be odd) with integrable representations of $\mathcal{U}_q(sl_{n+1}^{tor})$. We expect these modules to satisfy the definition of extremal loop weight modules.

Let us outline the main steps of the construction of level 0 extremal fundamental loop weight modules associated to the monomial crystals $\mathcal{M}(Y_{\ell,0}Y_{0,\ell}^{-1})$. This construction is based on the combinatorial study of these crystals. The cyclic symmetry of the Dynkin diagram of type $A_n^{(1)}$ has a counterpart at the level of crystals. Actually, these symmetry properties are already known for the $\mathcal{U}_q(sl_{n+1})$ -crystals of finite type, and are given by the existence of promotion operators (see [2, 16, 42, 44, 45] and references therein). Here we introduce promotion operators for the affinized $\mathcal{U}_q(\hat{sl}_{n+1})$ -crystals and for the level 0 extremal fundamental weight crystals $\mathcal{B}(\varpi_\ell)$ ($1 \leq \ell \leq n$). We explicit these operators in the monomial realizations $\mathcal{M}(Y_{\ell,0}Y_{0,\ell}^{-1})$. In particular, we get a new description of these monomial crystals. Furthermore we introduce the notion of closed monomial set, related to the theory of q -characters and crystal bases. It gives a necessary condition for a set to be the set of ℓ -weights of an integrable representation. Finally, we determine when $\mathcal{M}(Y_{\ell,0}Y_{0,\ell}^{-1})$ is closed, using promotion operators: this is closed if and only if $\ell = 1$ or $\ell = r + 1$ (Proposition 3.23).

When $\mathcal{M}(Y_{\ell,0}Y_{0,\ell}^{-1})$ is closed, we construct an associated representation of $\mathcal{U}_q(sl_{n+1}^{tor})$ whose q -character is the sum of monomials occurring in $\mathcal{M}(Y_{\ell,0}Y_{0,\ell}^{-1})$ with multiplicity one. The construction goes as follows: we endow the vector space freely generated by monomials occurring in $\mathcal{M}(Y_{\ell,0}Y_{0,\ell}^{-1})$ with an action of $\mathcal{U}_q(sl_{n+1}^{tor})$ by pasting together

some finite-dimensional representations of the quantum affine algebra $\mathcal{U}_q(\hat{sl}_{n+1})$. We check that the relations of the quantum toroidal algebra are satisfied on this module using the promotion operators at the level of crystals. Moreover, these representations satisfy the definition of extremal loop weight modules of extremal ℓ -weights $Y_{\ell,0}Y_{0,\ell}^{-1}$. They are irreducible, isomorphic to the level 0 fundamental extremal representations $V(\varpi_\ell)$ as modules over the horizontal subalgebra of $\mathcal{U}_q(sl_{n+1}^{tor})$. Furthermore formulas of the action of the quantum toroidal algebra are given in terms of constants of the associated crystal. By specializing q at roots of unity ϵ , we get finite-dimensional representations of $\mathcal{U}_\epsilon(sl_{n+1}^{tor})$.

When the monomial crystal $\mathcal{M}(Y_{\ell,0}Y_{0,\ell}^{-1})$ is not closed, there is no representation of $\mathcal{U}_q(sl_{n+1}^{tor})$ whose q -character is the sum of monomials occurring in $\mathcal{M}(Y_{\ell,0}Y_{0,\ell}^{-1})$. In fact, some monomials which should as predicted by the theory of q -characters are missing from the crystal. The idea is to consider instead of $\mathcal{M}(Y_{\ell,0}Y_{0,\ell}^{-1})$ a closed crystal containing it and to apply the preceding methods for this crystal. We treat an example of such a construction: we define a representation of $\mathcal{U}_q(sl_4^{tor})$ generated by a vector of ℓ -weight $Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1}$ and we check that it satisfies the definition of extremal loop weight modules.

Let us now describe briefly the organization of this paper.

In Section 2 we recall the definitions of quantum affine algebras $\mathcal{U}_q(\hat{sl}_{n+1})$ and quantum toroidal algebras $\mathcal{U}_q(sl_{n+1}^{tor})$ and we briefly review their representation theory. In particular one defines the extremal weight modules and the extremal loop weight modules. Section 3 is devoted to the study of monomial crystals. We recall its definition and we introduce the notion of closed monomial set (Definition 3.6). We define promotion operators for the level 0 fundamental extremal weight crystals. As a consequence, we determine when $\mathcal{M}(Y_{\ell,0}Y_{0,\ell}^{-1})$ is closed (Proposition 3.23). In Section 4 we construct a new family of representations of $\mathcal{U}_q(sl_{n+1}^{tor})$, called level 0 extremal fundamental loop weight modules, when n is odd and $\mathcal{M}(Y_{\ell,0}Y_{0,\ell}^{-1})$ is closed (Theorem 4.1). We check that these representations satisfy the definition of extremal loop weight modules for $\mathcal{U}_q(sl_{n+1}^{tor})$ (Theorem 4.6) and we give formulas for the action (Theorem 4.10). We get finite-dimensional representations of $\mathcal{U}_\epsilon(sl_{n+1}^{tor})$ by specializing the quantum parameter q at roots of unity ϵ (Theorem 4.15). In Section 5 we treat an example where the considered monomial crystal is not closed. We construct a representation of $\mathcal{U}_q(sl_4^{tor})$ associated to (a closed crystal containing) the monomial crystal $\mathcal{M}(Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})$. In Section 6 other possible developments and applications of these results are discussed.

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2. BACKGROUND

We recall the main definitions and general properties about the representation theory of quantum affine algebras and quantum toroidal algebras of type A.

2.1. Cartan matrix. Let $C = (C_{i,j})_{0 \leq i,j \leq n}$ be a Cartan matrix of type $A_n^{(1)}$,

$$C = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & -1 \\ -1 & 2 & \ddots & \ddots & & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & & \ddots & \ddots & 2 & -1 \\ -1 & 0 & \cdots & 0 & -1 & 2 \end{pmatrix}.$$

Set $I = \{0, \dots, n\}$ and $I_0 = \{1, \dots, n\}$. In particular, $(C_{i,j})_{i,j \in I_0}$ is the Cartan matrix of finite type A_n . In the following, I will be often identified with the set $\mathbb{Z}/(n+1)\mathbb{Z}$. Consider the vector space of dimension $n+2$

$$\mathfrak{h} = \mathbb{Q}h_0 \oplus \mathbb{Q}h_1 \oplus \cdots \oplus \mathbb{Q}h_n \oplus \mathbb{Q}d$$

and the linear functions α_i (the simple roots), Λ_i (the fundamental weights) on \mathfrak{h} given by $(i, j \in I)$

$$\begin{aligned} \alpha_i(h_j) &= C_{j,i}, & \alpha_i(d) &= \delta_{0,i}, \\ \Lambda_i(h_j) &= \delta_{i,j}, & \Lambda(d) &= 0. \end{aligned}$$

Denote by $\Pi = \{\alpha_0, \dots, \alpha_n\} \subset \mathfrak{h}^*$ the set of simple roots and $\Pi^\vee = \{h_0, \dots, h_n\} \subset \mathfrak{h}$ the set of simple coroots. Let $P = \{\lambda \in \mathfrak{h}^* \mid \lambda(h_i) \in \mathbb{Z} \text{ for any } i \in I\}$ be the weight lattice and $P^+ = \{\lambda \in P \mid \lambda(\alpha_i^\vee) \geq 0 \text{ for any } i \in I\}$ the semigroup of dominant weights. Let $Q = \bigoplus_{i \in I} \mathbb{Z}\alpha_i \subset P$ (the root lattice) and $Q^+ = \sum_{i \in I} \mathbb{N}\alpha_i \subset Q$. For $\lambda, \mu \in \mathfrak{h}^*$, write $\lambda \geq \mu$ if $\lambda - \mu \in Q^+$.

Denote by \hat{W} the affine Weyl group: it is the subgroup of $GL(\mathfrak{h}^*)$ generated by the simple reflections $s_i \in GL(\mathfrak{h}^*)$ defined by $s_i(\lambda) = \lambda - \lambda(h_i)\alpha_i$ ($i \in I$). The Weyl group of finite type W is the subgroup of \hat{W} generated by the s_i with $i \in I_0$.

Let $c = h_0 + \cdots + h_n$ and $\delta = \alpha_0 + \cdots + \alpha_n$. We have

$$\{\omega \in P \mid \omega(h_i) = 0 \text{ for all } i \in I\} = \mathbb{Q}\delta \cap P.$$

Put $P_{\text{cl}} = P/(\mathbb{Q}\delta \cap P)$ and denote by $\text{cl} : P \rightarrow P_{\text{cl}}$ the canonical projection. Let denote by $P^0 = \{\lambda \in P \mid \lambda(c) = 0\}$ the set of level 0 weights.

2.2. Quantum affine algebra $\mathcal{U}_q(\hat{\mathfrak{sl}}_{n+1})$. In this article $q \in \mathbb{C}^*$ is not a root of unity and is fixed. For $l \in \mathbb{Z}, r \geq 0, m \geq m' \geq 0$ consider the following Laurent polynomials in $\mathbb{Z}[q^\pm]$:

$$[l]_q = \frac{q^l - q^{-l}}{q - q^{-1}} \in \mathbb{Z}[q^\pm], \quad [r]_q! = [r]_q[r-1]_q \cdots [1]_q, \quad \begin{bmatrix} m \\ m' \end{bmatrix}_q = \frac{[m]_q!}{[m-m']_q![m']_q!}.$$

Definition 2.1. *The quantum affine algebra $\mathcal{U}_q(\hat{sl}_{n+1})$ is the \mathbb{C} -algebra with generators k_h ($h \in \mathfrak{h}$), x_i^\pm ($i \in I$) and relations*

$$\begin{aligned} (1) \quad & k_h k_{h'} = k_{h+h'}, \quad k_0 = 1, \\ (2) \quad & k_h x_j^\pm k_{-h} = q^{\pm \alpha_j(h)} x_j^\pm, \\ (3) \quad & [x_i^+, x_j^-] = \delta_{i,j} \frac{k_i - k_i^{-1}}{q - q^{-1}}, \\ (4) \quad & (x_i^\pm)^{(2)} x_{i+1}^\pm - x_i^\pm x_{i+1}^\pm x_i^\pm + x_{i+1}^\pm (x_i^\pm)^{(2)} = 0. \end{aligned}$$

where we use the notation $k_i^{\pm 1} = k_{\pm h_i}$ and for all $r \geq 0$ we set $(x_i^\pm)^{(r)} = \frac{(x_i^\pm)^r}{[r]_q!}$. One defines a Hopf algebra structure on $\mathcal{U}_q(\hat{sl}_{n+1})$ by setting

$$\begin{aligned} \Delta(k_h) &= k_h \otimes k_h, \\ \Delta(x_i^+) &= x_i^+ \otimes 1 + k_i^+ \otimes x_i^+, \quad \Delta(x_i^-) = x_i^- \otimes k_i^- + 1 \otimes x_i^-. \end{aligned}$$

Let $\mathcal{U}_q(\hat{sl}_{n+1})'$ be the subalgebra of $\mathcal{U}_q(\hat{sl}_{n+1})$ generated by x_i^\pm and k_h ($h \in \sum \mathbb{Q}h_i$). This has P_{cl} as a weight lattice.

For $J \subset I$ denote by $\mathcal{U}_q(\hat{sl}_{n+1})_J$ the subalgebra of $\mathcal{U}_q(\hat{sl}_{n+1})$ generated by the x_i^\pm, k_{ph_i} for $i \in J, p \in \mathbb{Q}$. If $J = I_0$, $\mathcal{U}_q(\hat{sl}_{n+1})_{I_0}$ is the quantum group of finite type associated to the simple Lie algebra sl_{n+1} , also denoted by $\mathcal{U}_q(sl_{n+1})$. In particular, a $\mathcal{U}_q(\hat{sl}_{n+1})$ -module has a structure of $\mathcal{U}_q(sl_{n+1})$ -module. If $J = \{i\}$ with $i \in I$, $\mathcal{U}_q(\hat{sl}_{n+1})_J$ is isomorphic to $\mathcal{U}_q(sl_2)$ and denoted by \mathcal{U}_i . So a $\mathcal{U}_q(\hat{sl}_{n+1})$ -module has also a structure of $\mathcal{U}_q(sl_2)$ -module.

Let $\mathcal{U}_q(\hat{sl}_{n+1})^+$ (resp. $\mathcal{U}_q(\hat{sl}_{n+1})^-$, $\mathcal{U}_q(\mathfrak{h})$) be the subalgebra of $\mathcal{U}_q(\hat{sl}_{n+1})$ generated by the x_i^+ (resp. the x_i^- , resp the k_h). We have a triangular decomposition of $\mathcal{U}_q(\hat{sl}_{n+1})$ (see [35]):

Theorem 2.2. *We have an isomorphism of vector spaces*

$$\mathcal{U}_q(\hat{sl}_{n+1}) \simeq \mathcal{U}_q(\hat{sl}_{n+1})^- \otimes \mathcal{U}_q(\mathfrak{h}) \otimes \mathcal{U}_q(\hat{sl}_{n+1})^+.$$

2.3. Representations of $\mathcal{U}_q(\hat{sl}_{n+1})$. For V a representation of $\mathcal{U}_q(\hat{sl}_{n+1})$ and $\nu \in P$, the weight space V_ν of V is

$$V_\nu = \{v \in V \mid k_h \cdot v = q^{\nu(h)} v, \forall h \in \mathfrak{h}\}.$$

For $\lambda \in P$, a representation V is said to be of highest weight λ if there is $v \in V_\lambda$ such that for all $i \in I$, $x_i^+ \cdot v = 0$ and $\mathcal{U}_q(\hat{sl}_{n+1}) \cdot v = V$. Furthermore there is a unique simple highest weight module $V(\lambda)$ of highest weight λ .

Definition 2.3. *A representation V is said to be integrable if*

- (1) *it admits a weight space decomposition $V = \bigoplus_{\nu \in P} V_\nu$,*
- (2) *all the x_i^\pm ($i \in I$) are locally nilpotent.*

Remark 2.4. *This definition is different of the one given in [23]. In fact it is required in addition in [23] that the representation V satisfies*

- (3) V_ν is finite-dimensional for any $\nu \in P$,
- (4) $V_{\nu \pm N\alpha_i} = \{0\}$ for all $\nu \in P$, $N \gg 0$, $i \in I$.

These conditions are implied by the previous ones for the highest weight modules.

Theorem 2.5. [35] *The simple highest weight module $V(\lambda)$ of highest weight λ is integrable if and only if λ is dominant.*

For an integrable representation V of $\mathcal{U}_q(\hat{sl}_{n+1})$ with finite-dimensional weight spaces, one defines the usual character

$$\chi(V) = \sum_{\nu \in P} \dim(V_\nu) e(\nu) \in \prod_{\nu \in P} \mathbb{Z} e(\nu).$$

Similar definitions hold for the quantum group $\mathcal{U}_q(sl_{n+1})$. In this case, the simple integrable representations $V(\lambda)$ ($\lambda \in P^+$) are finite-dimensional (see [35, 43]). Let \mathcal{C} be the category of finite-dimensional representations of $\mathcal{U}_q(sl_{n+1})$ and \mathcal{R} its Grothendieck ring.

Theorem 2.6. [35, 43] *The category \mathcal{C} is a semi-simple tensor category and the simple objects of \mathcal{C} are the $(V(\lambda))_{\lambda \in P^+}$. Furthermore χ induces a ring morphism*

$$\chi : \mathcal{R} \rightarrow \bigoplus_{\nu \in P} \mathbb{Z} e(\nu)$$

where the product at the right hand side is defined by $e(\mu)e(\nu) = e(\mu + \nu)$.

We do not recall here the theory of crystal bases of quantum groups, we just refer to [31, 32, 33]. When we want to distinguish crystals of $\mathcal{U}_q(\hat{sl}_{n+1})$, $\mathcal{U}_q(sl_{n+1})_J$ with $J \subset I$ and $\mathcal{U}_q(\hat{sl}_{n+1})'$, we call it respectively a P -crystal or I -crystal, a J -crystal and a P_{cl} -crystal.

2.4. Extremal weight modules. In this section we recall the definition and some properties of extremal weight modules for the quantum affine algebra $\mathcal{U}_q(\hat{sl}_{n+1})$ given by Kashiwara [31, 33]. All of these hold for general quantum Kac-Moody algebras and in particular for $\mathcal{U}_q(sl_{n+1})$.

Definition 2.7. *For an integrable $\mathcal{U}_q(\hat{sl}_{n+1})$ -module V and $\lambda \in P$, a vector $v \in V_\lambda$ is called extremal of weight λ if there are vectors $\{v_w\}_{w \in \hat{W}}$ such that $v_{Id} = v$ and*

$$x_i^\pm \cdot v_w = 0 \text{ if } \pm w(\lambda)(h_i) \geq 0 \text{ and } (x_i^\mp)^{(\pm w(\lambda)(h_i))} \cdot v_w = v_{s_i(w)}.$$

Note that if the vector v is extremal of weight λ , then for $w \in \hat{W}$, v_w is extremal of weight $w(\lambda)$. We denote it by $S_w v$, or simply by $S_i v$ if $w = s_i$ is a simple reflexion. Set $\hat{W} \cdot v = \{S_w v | w \in \hat{W}\}$.

The notion of extremal elements in a crystal \mathcal{B} can be defined in the same way.

Definition 2.8. *For $\lambda \in P$, the extremal weight module $V(\lambda)$ of extremal weight λ is the $\mathcal{U}_q(\hat{sl}_{n+1})$ -module generated by a vector v_λ with the defining relations that v_λ is extremal of weight λ .*

Example 2.9. *If λ is dominant, $V(\lambda)$ is the simple highest weight module of highest weight λ .*

Theorem 2.10. [31] *For $\lambda \in P$, the module $V(\lambda)$ is integrable and has a crystal basis $\mathcal{B}(\lambda)$.*

Set $\lambda = \varpi_\ell$, where $1 \leq \ell \leq n$ and ϖ_ℓ is the level 0 fundamental weight $\varpi_\ell = \Lambda_\ell - \Lambda_0$.

Theorem 2.11. [33] *Let $1 \leq \ell \leq n$.*

- (1) *$V(\varpi_\ell)$ is an irreducible $\mathcal{U}_q(\hat{sl}_{n+1})$ -module.*
- (2) *Any non-zero integrable $\mathcal{U}_q(\hat{sl}_{n+1})$ -module generated by an extremal weight vector of weight ϖ_ℓ is isomorphic to $V(\varpi_\ell)$.*

Let w be an element of \hat{W} such that $w(\varpi_\ell) = \varpi_\ell + \delta$. Such an element exists and is not unique (see [33]). There is a P_{cl} -automorphism z_ℓ of $V(\varpi_\ell)$ of weight δ , which sends v to v_w . Let us define the $\mathcal{U}_q(\hat{sl}_{n+1})'$ -module

$$W(\varpi_\ell) = V(\varpi_\ell)/(z_\ell - 1)V(\varpi_\ell).$$

Then we have

Theorem 2.12. [33] *Let $1 \leq \ell \leq n$.*

- (1) *$W(\varpi_\ell)$ is a finite-dimensional irreducible $\mathcal{U}_q(\hat{sl}_{n+1})'$ -module.*
- (2) *For any $\mu \in \text{wt}(V(\varpi_\ell))$,*

$$W(\varpi_\ell)_{\text{cl}(\mu)} \simeq V(\varpi_\ell)_\mu.$$

- (3) *$V(\varpi_\ell)$ is isomorphic to $W(\varpi_\ell)_{\text{aff}}$ as a $\mathcal{U}_q(\hat{sl}_{n+1})$ -module.*

Here M_{aff} is the affinization of an integrable $\mathcal{U}_q(\hat{sl}_{n+1})'$ -module M : this is the $\mathcal{U}_q(\hat{sl}_{n+1})$ -module defined by

$$M_{\text{aff}} = \bigoplus_{\nu \in P} M_{\text{cl}(\nu)},$$

with the obvious action of x_i^\pm . Note also that we have an isomorphism of $\mathcal{U}_q(\hat{sl}_{n+1})'$ -modules

$$M_{\text{aff}} \simeq \mathbb{C}[z, z^{-1}] \otimes M$$

where x_i^\pm act on the right hand side by $z^{\pm \delta_{i,0}} x_i^\pm$. In the same way one defines the affinization \mathcal{B}_{aff} of a P_{cl} -crystal \mathcal{B} . For an integrable $\mathcal{U}_q(\hat{sl}_{n+1})'$ -module M with associated P_{cl} -crystal \mathcal{B} , its affinization M_{aff} has a crystal \mathcal{B}_{aff} .

2.5. Quantum toroidal algebra $\mathcal{U}_q(sl_{n+1}^{\text{tor}})$. In this section, we recall the definition and the main properties of the quantum toroidal algebra $\mathcal{U}_q(sl_{n+1}^{\text{tor}})$ (without central charge) and its representations.

Definition 2.13. [20] *The quantum toroidal algebra $\mathcal{U}_q(sl_{n+1}^{\text{tor}})$ is the \mathbb{C} -algebra with generators $x_{i,r}^\pm$ ($i \in I, r \in \mathbb{Z}$), k_h ($h \in \mathfrak{h}$), $h_{i,m}$ ($i \in I, m \in \mathbb{Z} - \{0\}$) and the following relations ($i, j \in I, r, r', r_1, r_2 \in \mathbb{Z}, m \in \mathbb{Z} - \{0\}$)*

$$(5) \quad k_h k_{h'} = k_{h+h'}, \quad k_0 = 1, \quad [k_h, h_{j,m}] = 0, \quad [h_{i,m}, h_{j,m'}] = 0,$$

$$(6) \quad k_h x_{j,r}^\pm k_{-h} = q^{\pm \alpha_j(h)} x_{j,r}^\pm,$$

$$(7) \quad [h_{i,m}, x_{j,r}^\pm] = \pm \frac{1}{m} [mC_{i,j}]_q x_{j,m+r}^\pm,$$

$$(8) \quad [x_{i,r}^+, x_{j,r'}^-] = \delta_{ij} \frac{\phi_{i,r+r'}^+ - \phi_{i,r+r'}^-}{q - q^{-1}},$$

$$(9) \quad x_{i,r+1}^\pm x_{j,r'}^\pm - q^{\pm C_{ij}} x_{j,r'}^\pm x_{i,r+1}^\pm = q^{\pm C_{ij}} x_{i,r}^\pm x_{j,r'+1}^\pm - x_{j,r'+1}^\pm x_{i,r}^\pm,$$

$$(10) \quad \begin{aligned} & x_{i,r_1}^\pm x_{i,r_2}^\pm x_{i+1,r'}^\pm - (q + q^{-1}) x_{i,r_1}^\pm x_{i+1,r'}^\pm x_{i,r_2}^\pm + x_{i+1,r'}^\pm x_{i,r_1}^\pm x_{i,r_2}^\pm = \\ & -x_{i,r_2}^\pm x_{i,r_1}^\pm x_{i+1,r'}^\pm + (q + q^{-1}) x_{i,r_2}^\pm x_{i+1,r'}^\pm x_{i,r_1}^\pm - x_{i+1,r'}^\pm x_{i,r_2}^\pm x_{i,r_1}^\pm, \end{aligned}$$

where for all $i \in I$ and $m \in \mathbb{Z}$, $\phi_{i,m}^\pm \in \mathcal{U}_q(sl_{n+1}^{tor})$ is determined by the formal power series in $\mathcal{U}_q(sl_{n+1}^{tor})[[z]]$ (resp. in $\mathcal{U}_q(sl_{n+1}^{tor})[[z^{-1}]]$)

$$\sum_{m \geq 0} \phi_{i,\pm m}^\pm z^{\pm m} = k_i^{\pm 1} \exp \left(\pm (q - q^{-1}) \sum_{m' \geq 1} h_{i,\pm m'} z^{\pm m'} \right)$$

and $\phi_{i,m}^+ = 0$ for $m < 0$, $\phi_{i,m}^- = 0$ for $m > 0$.

There is an algebra morphism $\mathcal{U}_q(\hat{sl}_{n+1}) \rightarrow \mathcal{U}_q(sl_{n+1}^{tor})$ defined by $(h \in \mathfrak{h}, i \in I)$ $k_h \mapsto k_h$, $x_i^\pm \mapsto x_{i,0}^\pm$. Its image is called the horizontal quantum affine subalgebra of $\mathcal{U}_q(sl_{n+1}^{tor})$ and is denoted by $\mathcal{U}_q^h(sl_{n+1}^{tor})$. In particular, a $\mathcal{U}_q(sl_{n+1}^{tor})$ -module has also a structure of a $\mathcal{U}_q(\hat{sl}_{n+1})$ -module.

For $i \in I$, the subalgebra $\hat{\mathcal{U}}_i$ generated by the $x_{i,r}^\pm, h_{i,m}, k_{ph_i}$ ($r \in \mathbb{Z}, m \in \mathbb{Z} \setminus \{0\}, p \in \mathbb{Q}$) is isomorphic to $\mathcal{U}_q(\hat{sl}_2)$.

For all $j \in I$, set $I_j = I - \{j\}$ and define the subalgebra $\mathcal{U}_q^{v,j}(sl_{n+1}^{tor})$ of $\mathcal{U}_q(sl_{n+1}^{tor})$ generated by the $x_{i,r}^\pm, k_h, h_{i,m}$ ($i \in I_j, r \in \mathbb{Z}, h \in \bigoplus_{i \in I_j} \mathbb{Q} h_i, m \in \mathbb{Z} - \{0\}$). $\mathcal{U}_q^{v,0}(sl_{n+1}^{tor})$ is simply denoted by $\mathcal{U}_q^v(sl_{n+1}^{tor})$ and is called the vertical quantum affine subalgebra of $\mathcal{U}_q(sl_{n+1}^{tor})$. It is an untwisted quantum affine algebra isomorphic to $\mathcal{U}_q(\hat{sl}_{n+1})$ [3, 11]. Furthermore all the $\mathcal{U}_q^{v,j}(sl_{n+1}^{tor})$ for various $j \in I$ are isomorphic. In fact, let θ be the automorphism of the Dynkin diagram of type $A_n^{(1)}$ corresponding to the rotation such that $\theta(k) = k + 1$, where I is identified to the set $\mathbb{Z}/(n+1)\mathbb{Z}$. The isomorphism $\theta^{(j)}$ between $\mathcal{U}_q^v(sl_{n+1}^{tor})$ and $\mathcal{U}_q^{v,j}(sl_{n+1}^{tor})$ is simply given by sending $x_{i,r}^\pm, k_i^\pm, h_{i,m}$ to $x_{\theta^j(i),r}^\pm, k_{\theta^j(i)}^\pm, h_{\theta^j(i),m}$ respectively (where $i \in I_0, r \in \mathbb{Z}, m \in \mathbb{Z} - \{0\}$). If V is a $\mathcal{U}_q(\hat{sl}_{n+1})$ -module, we denote by $V^{(j)}$ the induced $\mathcal{U}_q^{v,j}(sl_{n+1}^{tor})$ -module.

We have a triangular decomposition of $\mathcal{U}_q(sl_{n+1}^{tor})$.

Theorem 2.14. [36, 39] *We have an isomorphism of vector spaces*

$$\mathcal{U}_q(sl_{n+1}^{tor}) \simeq \mathcal{U}_q(sl_{n+1}^{tor})^- \otimes \mathcal{U}_q(\hat{\mathfrak{h}}) \otimes \mathcal{U}_q(sl_{n+1}^{tor})^+,$$

where $\mathcal{U}_q(sl_{n+1}^{tor})^\pm$ (resp. $\mathcal{U}_q(\hat{\mathfrak{h}})$) is generated by the $x_{i,m}^\pm$ (resp. the k_h , the $h_{i,r}$).

2.6. Representations of $\mathcal{U}_q(sl_{n+1}^{tor})$.

Definition 2.15. A representation of $\mathcal{U}_q(sl_{n+1}^{tor})$ is said to be integrable if it is integrable as a $\mathcal{U}_q(\hat{sl}_{n+1})$ -module.

Definition 2.16. A representation V of $\mathcal{U}_q(sl_{n+1}^{tor})$ is said to be of ℓ -highest weight if there is $v \in V$ such that

- (1) $V = \mathcal{U}_q(sl_{n+1}^{tor})^- \cdot v$,
- (2) $\mathcal{U}_q(\hat{\mathfrak{h}}) \cdot v = \mathbb{C}v$,
- (3) for any $i \in I, m \in \mathbb{Z}$, $x_{i,m}^+ \cdot v = 0$.

For $\gamma \in \mathcal{U}_q(\hat{\mathfrak{h}}) \rightarrow \mathbb{C}$ an algebra morphism, by Theorem 2.14 we have a corresponding Verma module $M(\gamma)$ and a simple representation $V(\gamma)$ which are ℓ -highest weight. Then we have:

Theorem 2.17. [36, 39] The simple integrable representations $V(\gamma)$ of $\mathcal{U}_q(sl_{n+1}^{tor})$ are integrable if there is $(\lambda, (P_i)_{1 \leq i \leq n}) \in P \times (1 + u\mathbb{C}[u])^n$ satisfying $\gamma(k_h) = q^{\lambda(h)}$ and for $i \in I$ the relation in $\mathbb{C}[[z]]$ (resp. in $\mathbb{C}[[z^{-1}]]$)

$$\gamma(\phi_i^\pm(z)) = q^{\pm \deg(P_i)} \frac{P_i(zq^{-1})}{P_i(zq)}.$$

The polynomials P_i are called Drinfeld polynomials and the representation $V(\gamma)$ is then denoted by $V((P_i)_{i \in I})$.

For all $k \geq 0$, $a \in \mathbb{C}^*$ and $1 \leq \ell \leq n$, the Kirillov-Reshetikhin module $W_{k,a}^{(\ell)}$ is the simple integrable representation of weight $k\Lambda_\ell$ with the n -tuple

$$P_i(u) = \begin{cases} (1 - ua)(1 - uaq^2) \cdots (1 - uaq^{2(k-1)}) & \text{for } i = \ell, \\ 1 & \text{for } i \neq \ell. \end{cases}$$

The representations $V_\ell(a) = W_{1,a}^{(\ell)}$ are called fundamental modules.

Consider an integrable representation V of $\mathcal{U}_q(sl_{n+1}^{tor})$. As the subalgebra $\mathcal{U}_q(\hat{\mathfrak{h}})$ is commutative, we have a decomposition of the weight spaces V_ν in simultaneous generalized eigenspaces

$$V_\nu = \bigoplus_{(\nu, \gamma)} V_{(\nu, \gamma)},$$

where $V_{(\nu, \gamma)} = \{x \in V / \exists p \in \mathbb{N}, \forall i \in \{1, \dots, n\}, \forall m \geq 0, (\phi_{i, \pm m}^\pm - \gamma_{i, \pm m}^\pm)^p \cdot x = 0\}$ is finite-dimensional. If $V_{(\nu, \gamma)} \neq \{0\}$, (ν, γ) is called an ℓ -weight of V .

Definition 2.18. [19, 21, 39] The q -character of an integrable representation V of $\mathcal{U}_q(sl_{n+1}^{tor})$ with finite-dimensional ℓ -weight spaces is defined by the formal sum

$$\chi_q(V) = \sum_{(\nu, \gamma) \in P \times \mathbb{C}^\mathbb{Z}} \dim(V_{(\nu, \gamma)}) e(\nu, \gamma).$$

Furthermore we have the equality

$$\chi(\text{Res}(V)) = \beta(\chi_q(V)),$$

where $\text{Res}(V)$ is the restricted $\mathcal{U}_q(\hat{sl}_{n+1})$ -module obtained from V and

$$\beta : \prod_{(\nu, \gamma) \in P \times \mathbb{C}^\mathbb{Z}} \mathbb{Z}e(\nu, \gamma) \rightarrow \prod_{\nu \in P} \mathbb{Z}e(\nu)$$

is \mathbb{Z} -linear such that $\beta(e(\nu, \gamma)) = e(\nu)$ for all $(\nu, \gamma) \in P \times \mathbb{C}^\mathbb{Z}$.

Proposition 2.19. [19, 21, 39] *Let V be an integrable representation of $\mathcal{U}_q(sl_{n+1}^{tor})$. An ℓ -weight (ν, γ) of V satisfies the properties*

- (1) *there exist polynomials $Q_i(z), R_i(z) \in \mathbb{C}[z]$ ($i \in I$) of constant term 1 such that in $\mathbb{C}[[z]]$ (resp. in $\mathbb{C}[[z^{-1}]]$):*

$$\sum_{m \geq 0} \gamma_{i, \pm m}^\pm z^{\pm m} = q^{\deg(Q_i) - \deg(R_i)} \frac{Q_i(zq^{-1})R_i(zq)}{Q_i(zq)R_i(zq^{-1})}.$$

- (2) *there exist $\omega \in P^+, \alpha \in Q^+$ satisfying $\nu = \omega - \alpha$.*

At more, if V has a finite composition serie $L_0 = \{0\} \subset L_1 \subset L_2 \subset \dots \subset L_k = V$ such that $L_{j+1}/L_j \simeq V((P_i^j)_{i \in I})$ where the roots of P_i^j are in $q^\mathbb{Z}$ for all $i \in I, 0 \leq j \leq k-1$, then

- (3) *the zeros of the polynomials $Q_i(z), R_i(z)$ are in $q^\mathbb{Z}$.*

If V is a Kirillov-Reshetikhin module, one reduces to the case where the defining parameter a is in $q^\mathbb{Z}$ by twisting the action by the automorphisms τ_b of $\mathcal{U}_q(sl_{n+1}^{tor})$ given by ($b \in \mathbb{C}^*$)

$$\tau_b(x_{i,r}^\pm) = b^r x_{i,r}^\pm, \quad \tau_b(h_{i,m}^\pm) = b^m h_{i,m}^\pm, \quad \tau_b(k_i^\pm) = k_i^\pm.$$

Consider formal variables $Y_{i,l}^\pm, e^\nu$ ($i \in I, l \in \mathbb{Z}, \nu \in P$) and let A be the group of monomials of the form $m = e^{\omega(m)} \prod_{i \in I, l \in \mathbb{Z}} Y_{i,l}^{u_{i,l}(m)}$ where $u_{i,l}(m) \in \mathbb{Z}, \omega(m) \in P$ are such that

$$\sum_{l \in \mathbb{Z}} u_{i,l}(m) = \omega(m)(\alpha_i^\vee).$$

For $m \in A$ and $i \in I$ we set $u_i(m) = \sum_{l \in \mathbb{Z}} u_{i,l}(m)$. For example, $Y_{i,l}^\pm e^{\pm \Lambda_i} \in A$ and $A_{i,l} = e^{\alpha_i} Y_{i,l-1} Y_{i,l+1} Y_{i-1,l}^{-1} Y_{i+1,l}^{-1} \in A$. A monomial m is said to be J -dominant ($J \subset I$) if for all $j \in J$ and $l \in \mathbb{Z}$ we have $u_{j,l}(m) \geq 0$. An I -dominant monomial is said to be dominant.

Let V be an integrable $\mathcal{U}_q(sl_{n+1}^{tor})$ -module such that for all ℓ -weight (ν, γ) of V , the roots of the associated polynomials $Q_i(z)$ and $R_i(z)$ are in $q^\mathbb{Z}$ for all $i \in I$. For (ν, γ) an ℓ -weight of V , one defines the monomial $m_{(\nu, \gamma)} = e^\nu \prod_{i \in I, l \in \mathbb{Z}} Y_{i,l}^{u_{i,l} - v_{i,l}}$ where

$$Q_i(z) = \prod_{l \in \mathbb{Z}} (1 - zq^l)^{u_{i,l}} \text{ and } R_i(z) = \prod_{l \in \mathbb{Z}} (1 - zq^l)^{v_{i,l}}.$$

We denote $V_{(\nu, \gamma)} = V_{m_{(\nu, \gamma)}}$. We rewrite the q -character of an integrable representation V with finite-dimensional ℓ -weight spaces by the formal sum

$$\chi_q(V) = \sum_m \dim(V_m) m \in \mathbb{Z}[[Y_{i,l}]]_{i \in I, l \in \mathbb{Z}}.$$

Let us denote by $\mathcal{M}(V)$ the set of monomials occurring in $\chi_q(V)$.

By this correspondence between ℓ -weights and monomials due to Frenkel-Reshetikhin [19], the I -tuple of Drinfeld polynomials with zeros in $q^{\mathbb{Z}}$ are identified with the dominant monomials. In particular for a dominant monomial m , one denotes by $V(m)$ the simple module of ℓ -highest weight m .

Similar results hold for the quantum affine algebra $\mathcal{U}_q(\hat{sl}_{n+1})$ due to Chari-Pressley [8]. In this case, the simple integrable representations $V((P_i)_{i \in I_0})$ are finite-dimensional. Furthermore the Kirillov-Reshetikhin modules $W_{k,a}^{(\ell)}$ ($\ell \in I_0, a \in \mathbb{C}^*, k \geq 0$) can be obtained from the $\mathcal{U}_q(sl_{n+1})$ -modules $V(k\Lambda_\ell)$ by evaluation morphisms $\text{ev}_a : \mathcal{U}_q(\hat{sl}_{n+1}) \rightarrow \mathcal{U}_q(sl_{n+1})$. They are isomorphic as $\mathcal{U}_q(sl_{n+1})$ -modules for all $a \in \mathbb{C}^*$. As consequences of these results, $W(\varpi_\ell)$ is isomorphic to a fundamental representation $V_\ell(a)$ for a special choice of the spectral parameter $a \in \mathbb{C}^*$ and $V_\ell(a)$ has a crystal basis.

Let \mathcal{C}_l be the category of finite-dimensional $\mathcal{U}_q(\hat{sl}_{n+1})'$ -modules (of type 1) and \mathcal{R}_l its Grothendieck ring. Recall that \mathcal{C}_l is an abelian monoidal category, not semi-simple, with simple objects the $V((P_i)_{i \in I_0})$ and \mathcal{R}_l is the polynomial ring over \mathbb{Z} in the classes $[V_\ell(a)]$ ($\ell \in I_0, a \in \mathbb{C}^*$) (see [8, 19]). As in [26], we consider $\mathcal{C}_{l,\mathbb{Z}}$ the full subcategory of \mathcal{C}_l whose objects V satisfy

for every composition factor S of V , the roots of the Drinfeld polynomials $(P_i(u))_{i \in I_0}$ belong to $q^{\mathbb{Z}}$.

This is also an abelian monoidal category, not semi-simple and the Grothendieck ring $\mathcal{R}_{l,\mathbb{Z}}$ of $\mathcal{C}_{l,\mathbb{Z}}$ is the subring of \mathcal{R}_l generated by the classes $[V_\ell(q^s)]$ with $\ell \in I_0, s \in \mathbb{Z}$ (see [17]).

Theorem 2.20. [19] χ_q induces a ring morphism $\chi_q : \mathcal{R}_{l,\mathbb{Z}} \rightarrow \mathbb{Z}[Y_{i,l}]_{i \in I_0, l \in \mathbb{Z}}$, called morphism of q -characters. Furthermore we have the following commutative diagram

$$\begin{array}{ccc} \mathcal{R}_{l,\mathbb{Z}} & \xrightarrow{\chi_q} & \mathbb{Z}[Y_{i,l}]_{i \in I_0, l \in \mathbb{Z}} \\ \text{Res} \downarrow & & \downarrow \beta \\ \mathcal{R} & \xrightarrow{\chi} & \bigoplus_{\nu \in P} \mathbb{Z}e(\nu) \end{array}$$

where the ring morphism $\text{Res} : \mathcal{R}_{l,\mathbb{Z}} \rightarrow \mathcal{R}$ is the restriction and $\beta : \mathcal{R}_l \rightarrow \mathcal{R}$ is defined by $\beta(m) = e(\omega(m))$ for all $m \in A$.

One does not have expressions of the q -character of a representation in general. But explicit formulas exist for the fundamental modules and Kirillov-Reshetikhin modules, given in terms of tableaux [25, 40].

2.7. Extremal loop weight modules. We recall the notion of extremal loop weight modules for $\mathcal{U}_q(sl_{n+1}^{\text{tor}})$. The main motivation is the construction of finite-dimensional representations of the quantum toroidal algebra as in the theory of Kashiwara, but at roots of unity in this case.

Definition 2.21. [23] *An extremal loop weight module of $\mathcal{U}_q(sl_{n+1}^{\text{tor}})$ is an integrable representation V such that there is an ℓ -weight vector $v \in V$ satisfying*

$$(1) \quad \mathcal{U}_q(sl_{n+1}^{\text{tor}}) \cdot v = V,$$

- (2) v is extremal for $\mathcal{U}_q^h(sl_{n+1}^{tor})$,
- (3) $\mathcal{U}_q^{v,j}(sl_{n+1}^{tor}) \cdot w$ is finite-dimensional for all $w \in V$ and $j \in I$.

Example 2.22. *If m is dominant, the simple ℓ -highest weight module $V(m)$ of ℓ -highest weight m is an extremal loop weight module.*

An example of such a representation which is neither of ℓ -highest weight nor of ℓ -lowest weight is given in [23]. The goal of this article is to construct a new family of extremal loop weight modules, called *level 0 extremal fundamental loop weight modules*.

3. STUDY OF THE MONOMIAL CRYSTALS $\mathcal{M}(Y_{\ell,0}Y_{0,\ell}^{-1})$

We will relate in our paper the monomial realizations $\mathcal{M}(Y_{\ell,0}Y_{0,\ell}^{-1})$ of level 0 extremal fundamental weight crystals $\mathcal{B}(\varpi_\ell)$ of $\mathcal{U}_q(\hat{sl}_{n+1})$ with integrable representations of $\mathcal{U}_q(sl_{n+1}^{tor})$. In this section, we study the combinatorics of these monomial realizations, the main point being the use of promotion operators for level 0 extremal fundamental weight crystals introduced below. This is the first step of the construction of integrable modules associated to $\mathcal{M}(Y_{\ell,0}Y_{0,\ell}^{-1})$.

In the first part, one gives the definition of monomial crystals \mathcal{M} [34, 40]. This definition holds when the considered Cartan matrix has no odd cycles. So it does not work for $\mathcal{U}_q(\hat{sl}_{n+1})$ when n is even. Following [27], we recall the monomial realization of level 0 extremal weight crystals and in particular of $\mathcal{B}(\varpi_\ell)$ ($1 \leq \ell \leq n$): it is isomorphic to the sub- $\mathcal{U}_q(\hat{sl}_{n+1})$ -crystal $\mathcal{M}(Y_{\ell,0}Y_{0,\ell}^{-1})$ of \mathcal{M} generated by the monomial $Y_{\ell,0}Y_{0,\ell}^{-1}$. Furthermore we define the notions of q -closed monomial set and of monomial set closed by the Kashiwara operators, respectively related to the theory of q -characters and to the combinatorics of crystals.

The monomial crystals $\mathcal{M}(Y_{\ell,0}Y_{0,\ell}^{-1})$ are already studied in [27]: the monomials occurring in these crystals are explicitly given when $n \in \mathbb{N}$ is odd and $1 \leq \ell \leq n$ and their automorphisms z_ℓ are described in terms of monomials. We recall these results in the second part.

In the third part, we introduce affinized promotion operators for the affinization of $\mathcal{U}_q(sl_{n+1})$ -crystals of finite type. The promotion operators were introduced in [45] for the Young tableaux realization of the finite $\mathcal{U}_q(sl_{n+1})$ -crystals $\mathcal{B}(k\Lambda_\ell)$ ($k \in \mathbb{N}^*$, $1 \leq \ell \leq n$) and studied in numerous papers (see [2, 16, 42, 44, 45] and references therein). It is the counterpart at the level of crystals of the cyclic symmetry of the Dynkin diagram of type $A_n^{(1)}$. The promotion operator on $\mathcal{B}(k\Lambda_\ell)$ induces combinatorially a P_{cl} -crystal structure which is isomorphic to the crystal basis of the finite-dimensional Kirillov-Reshetikhin module $\mathcal{B}(W(k\varpi_\ell))$ (see [30]). We extend it to an operator of the P -crystal $\mathcal{B}(k\Lambda_\ell)_{\text{aff}}$ (affinization of $\mathcal{B}(k\Lambda_\ell)$), called affinized promotion operator. As a consequence, we get promotion operators for the level 0 extremal fundamental weight crystals $\mathcal{B}(\varpi_\ell)$ ($1 \leq \ell \leq n$). We identify the promotion operator on $\mathcal{B}(\varpi_\ell)$ for its monomial realization $\mathcal{M}(Y_{\ell,0}Y_{0,\ell}^{-1})$: it is explicitly given in terms of monomials.

In the last part, we use the promotion operator of $\mathcal{M}(Y_{\ell,0}Y_{0,\ell}^{-1})$ to obtain a new description of it. In particular, we precise results given in [27] for these crystals. Furthermore we determine the $\ell \in I_0$ for which the $\mathcal{U}_q(\hat{sl}_{2r+2})$ -crystals $\mathcal{M}(Y_{\ell,0}Y_{0,\ell}^{-1})$ are closed (Proposition 3.23): this is the case if and only if $\ell = 1$ or $\ell = r + 1$.

3.1. Monomial crystals. In this section we define the monomial crystal \mathcal{M} for $\mathcal{U}_q(\hat{sl}_{n+1})$ when $n = 2r + 1$ is supposed to be odd, following [34, 40]. The monomial realizations of the crystals $\mathcal{B}(\lambda)$ with $\lambda \in P$, in particular of $\mathcal{B}(\varpi_\ell)$ ($1 \leq \ell \leq n$), are studied in [27, 34, 40]. We recall these results here. Finally we introduce new notions of q -closed monomial set and of monomial set closed by the Kashiwara operators.

From now on, we suppose C without odd cycle, i.e. there is a function $s : I \rightarrow \{0, 1\}$, $i \mapsto s_i$ such that $C_{i,j} = -1$ implies $s_i + s_j = 1$. This is only the case for $\mathcal{U}_q(\hat{sl}_{n+1})$ if $n = 2r + 1$ is odd ($r \geq 1$).

Consider the subgroup $\mathcal{M} \subset A$ defined by

$$\mathcal{M} = \{m \in A \mid u_{i,l} = 0 \text{ if } l \equiv s_i + 1 \pmod{2}\}.$$

Following [34, 40], let us define $\text{wt} : \mathcal{M} \rightarrow P$ and $\varepsilon_i, \varphi_i, p_i, q_i : \mathcal{M} \rightarrow \mathbb{Z} \cup \{\infty\} \cup \{-\infty\}$ for $i \in I$ ($m \in \mathcal{M}$)

$$\begin{aligned} \text{wt}(m) &= \omega(m), \\ \varphi_{i,L}(m) &= \sum_{l \leq L} u_{i,l}(m), \quad \varphi_i(m) = \max\{\varphi_{i,L}(m) \mid L \in \mathbb{Z}\} \geq 0, \\ \varepsilon_{i,L}(m) &= -\sum_{l \geq L} u_{i,l}(m), \quad \varepsilon_i(m) = \max\{\varepsilon_{i,L}(m) \mid L \in \mathbb{Z}\} \geq 0, \\ p_i(m) &= \max\{L \in \mathbb{Z} \mid \varepsilon_{i,L}(m) = \varepsilon_i(m)\} \\ &= \max\left\{L \in \mathbb{Z} \mid \sum_{l < L} u_{i,l}(m) = \varphi_i(m)\right\}, \\ q_i(m) &= \min\{L \in \mathbb{Z} \mid \varphi_{i,L}(m) = \varphi_i(m)\} \\ &= \max\left\{L \in \mathbb{Z} \mid -\sum_{l > L} u_{i,l}(m) = \varepsilon_i(m)\right\}. \end{aligned}$$

Then we define $\tilde{e}_i, \tilde{f}_i : \mathcal{M} \rightarrow \mathcal{M} \cup \{0\}$ for $i \in I$ by

$$\begin{aligned} \tilde{e}_i \cdot m &= \begin{cases} 0 & \text{if } \varepsilon_i(m) = 0, \\ mA_{i,p_i(m)-1} & \text{if } \varepsilon_i(m) > 0, \end{cases} \\ \tilde{f}_i \cdot m &= \begin{cases} 0 & \text{if } \varphi_i(m) = 0, \\ mA_{i,q_i(m)+1}^{-1} & \text{if } \varphi_i(m) > 0. \end{cases} \end{aligned}$$

Theorem 3.1. [34, 40] $(\mathcal{M}, \text{wt}, \varepsilon_i, \varphi_i, \tilde{e}_i, \tilde{f}_i)$ is a P -crystal, called the monomial crystal.

Remark 3.2. When the considered Dynkin diagram is not bipartite (this is the case in type $A_n^{(1)}$ when n is even), $(\mathcal{M}, \text{wt}, \varepsilon_i, \varphi_i, \tilde{e}_i, \tilde{f}_i)$ does not satisfy the axioms of crystal (see [34]). Another crystal structures are defined on (a subset of) A in [34]. But the monomials used are different with those occurring in the theory of q -characters of $\mathcal{U}_q(sl_{n+1}^{\text{tor}})$ -modules and it is not useful for what we will do in the next sections.

For $m \in \mathcal{M}$ denote by $\mathcal{M}(m)$ the connected subcrystal of \mathcal{M} generated by m . In the same way, $\mathcal{M}_J(m)$ ($J \subset I$) is the connected sub- J -crystal of $\mathcal{M}(m)$ generated by m .

For $p \in \mathbb{Z}$ and $\alpha \in \mathbb{Q}\delta \cap P$, let $\tau_{2p,\alpha}$ be the map $\tau_{2p,\alpha}: \mathcal{M} \rightarrow \mathcal{M}$ defined by

$$\tau_{2p,\alpha}(e^\lambda \prod Y_{i,n}^{u_{i,n}}) = e^{\lambda+\alpha} \prod Y_{i,n+2p}^{u_{i,n}}.$$

This is a P_{cl} -crystal automorphism of the crystal \mathcal{M} . We often omit α from the notation and denote simply τ_{2p} .

The following result was proved in [34, 40] when m is a dominant monomial and is generalized in [27] for general $m \in \mathcal{M}$.

Theorem 3.3. *For $m \in \mathcal{M}$, the crystal $\mathcal{M}(m)$ is isomorphic to a connected component of the crystal $\mathcal{B}(\lambda)$ of an extremal weight module for some $\lambda \in P$.*

We get monomial realizations of fundamental I_0 -crystals $\mathcal{B}(\Lambda_i)$. As these crystals are connected, we have

Proposition 3.4. [34, 40] *For all $n \in \mathbb{N}^*$, $i \in I_0$ and $k \in \mathbb{Z}$, the $\mathcal{U}_q(\mathfrak{sl}_{n+1})$ -crystals $\mathcal{M}(Y_{i,k})$ and $\mathcal{B}(\Lambda_i)$ are isomorphic.*

It was shown in [33] that the fundamental extremal crystals $\mathcal{B}(\varpi_\ell)$ are connected ($\ell \in I_0$). We have the following monomial realizations of them.

Theorem 3.5. [27] *Assume that n is odd and set $M = Y_{\ell,0} Y_{0,\ell}^{-1}$ for $\ell \in I_0$. Then M is extremal in \mathcal{M} and $\mathcal{M}(M) \simeq \mathcal{B}(\varpi_\ell)$.*

For $i \in I$, set $\Xi_i: \mathcal{M} \rightarrow \mathcal{M}$ the map sending the variables $Y_{j,*}^{\pm 1}$ to 1 if $j \neq i$ and $Y_{i,*}^{\pm 1}$ to themselves. An other map will be used below: $\Xi^i: \mathcal{M} \rightarrow \mathcal{M}$ which sends the variables $Y_{j,*}^{\pm 1}$ to themselves if $j \neq i$ and $Y_{i,*}^{\pm 1}$ to 1.

Definition 3.6. (1) *A set of monomials $\mathcal{S} \subset \mathcal{M}$ is said to be q -closed in the direction i ($i \in I$) if for all $m \in \mathcal{S}$ there exists a finite subset $\mathcal{S}_m \subset \mathcal{S}$ containing m and a sequence $(n_s)_{s \in \mathcal{S}_m}$ of positive integers such that $\Xi_i(\sum_{s \in \mathcal{S}_m} n_s \cdot s)$ is the q -character of a representation of $\hat{\mathcal{U}}_i$.*

(2) *A set of monomials \mathcal{S} is said to be J - q -closed ($J \subset I$), or simply q -closed if $J = I$, if \mathcal{S} is q -closed in the direction i for all $i \in J$.*

(3) *A set of monomials $\mathcal{S} \subset \mathcal{M}$ is said to be closed by the Kashiwara operators if the operators \tilde{e}_i, \tilde{f}_i preserve \mathcal{S} for all $i \in I$.*

(4) *A set of monomials $\mathcal{S} \subset \mathcal{M}$ which is q -closed and closed by the Kashiwara operators, is called a closed set.*

Remark 3.7. *The definition of q -closed set is inspired by the theory of q -characters and the Frenkel-Mukhin algorithm [17]. This notion extend naturally when q is specialized at roots of unity, by using the theory q -character at roots of unity [18].*

Let V be an integrable $\mathcal{U}_q(\mathfrak{sl}_{n+1}^{\text{tor}})$ -module such that for all ℓ -weight (ν, γ) of V , $V_{(\nu, \gamma)}$ is finite-dimensional and the roots of the associated polynomials $Q_i(z)$ and $R_i(z)$ are in $q^{\mathbb{Z}}$ for all $i \in I$. Then the monomial set $\mathcal{M}(V)$ is q -closed. Note that it is not necessary

that the Frenkel-Mukhin algorithm holds for V : the simple finite-dimensional $\mathcal{U}_q(\hat{sl}_3)$ -module $V(Y_{1,0}^2 Y_{2,3}) \simeq V(Y_{1,0} Y_{2,3}) \otimes V(Y_{1,0})$ considered in [26] does not satisfy the Frenkel-Mukhin algorithm and $\mathcal{M}(V(Y_{1,0}^2 Y_{2,3}))$ is q -closed.

In general, $\mathcal{M}(V)$ is not closed by the Kashiwara operators: for example the q -character of the $\mathcal{U}_q(\hat{sl}_2)$ -module $V(Y_2 Y_0^2)$ contains the monomial Y_0 but do not contains Y_2^{-1} . But this question is partially solved in [40]: it is question to endow the set of monomials occurring in the (t -analogue of) q -character of a finite-dimensional representation of $\mathcal{U}_q(\hat{sl}_{n+1})$ with a structure of crystal. Let $m \in \mathcal{M}$ be a dominant monomial and $S(m)$ its associated standard module (the standard modules are finite-dimensional $\mathcal{U}_q(\hat{sl}_{n+1})$ -modules defined in [39], also parametrized by dominant monomials). Let $\widetilde{\chi}_{q,t}(S(m))$ be the t -analogue of q -character of $S(m)$ introduced by Nakajima [40] and still denote by $\mathcal{M}(S(m))$ the set of monomials occurring in $\widetilde{\chi}_{q,t}(S(m))$. Then

Proposition 3.8. [40] *$\mathcal{M}(S(m))$ has a crystal structure which is isomorphic to a direct sum of crystals of highest weight modules.*

One can precise this result when the considered standard modules are fundamental modules of $\mathcal{U}_q(\hat{sl}_{n+1})$. In fact by tableaux sum expressions of the q -characters of fundamental modules, one shows in a combinatorial way that

Proposition 3.9. *Let $V(Y_{i,k})$ be a fundamental representation of $\mathcal{U}_q(\hat{sl}_{n+1})$, with $i \in I_0$ and $k \in \mathbb{Z}$. Then $\mathcal{M}(V(Y_{i,k}))$ has a crystal structure isomorphic to $\mathcal{M}(Y_{i,k})$. More precisely, monomial sets $\mathcal{M}(Y_{i,k})$ and $\mathcal{M}(V(Y_{i,k}))$ are equal and the crystal isomorphism is given by the identity map.*

Corollary 3.10. *For all $1 \leq i \leq n$ and $k \in \mathbb{Z}$, the $\mathcal{U}_q(sl_{n+1})$ -crystal $\mathcal{M}(Y_{i,k})$ is closed.*

Finally, let us give an example of monomial crystal which is not q -closed. Consider the $\mathcal{U}_q(sl_2)$ -crystal $\mathcal{M}(Y_4 Y_0)$

$$Y_4 Y_0 \rightarrow Y_6^{-1} Y_0 \rightarrow Y_6^{-1} Y_2^{-1}.$$

If $\mathcal{M}(Y_4 Y_0)$ is q -closed, it should contain $\mathcal{M}(V(Y_4 Y_0))$. This is not the case, the q -character of $V(Y_4 Y_0)$ being

$$\chi_q(V(Y_4 Y_0)) = Y_4 Y_0 + Y_6^{-1} Y_0 + Y_4 Y_2^{-1} + Y_6^{-1} Y_2^{-1}.$$

3.2. Description of the monomial crystal $\mathcal{M}(Y_{\ell,0} Y_{0,\ell}^{-1})$. Assume that $n = 2r + 1$ is odd with $r \geq 1$. The monomial crystals $\mathcal{M}(Y_{\ell,0} Y_{0,\ell}^{-1})$ are studied in [27, Section 4]: the monomials occurring in these crystals are explicitly described and the automorphisms z_ℓ are given in terms of monomials. We recall these results here.

Assume that $\ell \leq r + 1$ (the case $\ell > r + 1$ can be obtain from these cases by applying a diagram automorphism).

One defines the monomials

$$\boxed{k}_p = Y_{k-1,p+k}^{-1} Y_{k,p+k-1} \quad \text{for } 1 \leq k \leq n + 1, p \in \mathbb{Z}.$$

with $Y_{n+1,p} = Y_{0,p}$ by convention. Set $M_0 = Y_{\ell,0}Y_{0,\ell}^{-1}$ and

$$\begin{aligned} M_j &= Y_{\ell,2j}Y_{0,n-\ell+1+2j}^{-1}Y_{j,\ell+j}^{-1}Y_{j,n-\ell+1+j} \\ &= \left(\boxed{1}_{n-\ell+2j} \boxed{2}_{n-\ell+2j-2} \cdots \boxed{j}_{n-\ell+2} \right) \times \left(\boxed{j+1}_{\ell-1} \boxed{j+2}_{\ell-3} \cdots \boxed{\ell}_{1-\ell+2j} \right) \\ &= \prod_{p=1}^j \boxed{p}_{n-\ell-2p+2j+2} \times \prod_{p=j+1}^{\ell} \boxed{p}_{\ell+1-2p+2j} \end{aligned}$$

with $0 \leq j \leq \ell$. In particular, $M_\ell = Y_{\ell,n+1}Y_{0,n+1+\ell}^{-1} = \tau_{n+1}(M_0)$ and $M_1 = \tau_2(M_0)$ for $\ell = r+1$. One defines other monomials in the following way: for $j \in \mathbb{Z}$ and a Young tableau of shape (ℓ) $T = (1 \leq i_1 < i_2 < \cdots < i_\ell \leq n+1)$ we set

$$(11) \quad m_{T;j} = \prod_{p=1}^j \boxed{i_p}_{n-\ell-2p+2j+2} \times \prod_{p=j+1}^{\ell} \boxed{i_p}_{\ell+1-2p+2j} \quad \text{for } 0 \leq j \leq \ell-1,$$

and $m_{T;j+\ell} = \tau_{n+1}m_{T;j}$.

By Theorem 3.5, $\mathcal{M}(M_0)$ and $\mathcal{B}(\varpi_\ell)$ are isomorphic as P -crystals. Furthermore

Proposition 3.11. [27]

- (1) $\mathcal{M}_{I_0}(M_j)$ consists of $m_{T;j}$ for various sequences T .
- (2) The map $\sigma : \mathcal{M}(Y_{\ell,0}Y_{0,\ell}^{-1}) \rightarrow \mathcal{M}(Y_{\ell,0}Y_{0,\ell}^{-1})$ defined by $\sigma(m_{T;j}) = m_{T;j+1}$ is a P_{cl} -crystal automorphism equal to z_ℓ^{-1} .
- (3) We have the equality of I_0 -crystals

$$(12) \quad \mathcal{M}(Y_{\ell,0}Y_{0,\ell}^{-1}) = \bigsqcup_{k \in \mathbb{Z}} \tau_{n+1}^k \left(\bigsqcup_{j=0}^n \mathcal{M}_{I_0}(M_j) \right).$$

- (4) The Kashiwara operators \tilde{e}_i, \tilde{f}_i are described in terms of tableaux: for $i \neq 0$ we have $\tilde{e}_i \cdot m_{T;j} = m_{T';j}$ or 0. Here T' is obtained from T by replacing $i+1$ by i . If it is not possible (i.e. when we have both $i+1$ and i in T), then it is zero. Similarly $\tilde{f}_i \cdot m_{T;j} = m_{T'';j}$ or 0, where T'' is given by replacing i by $i+1$. For the action of \tilde{e}_0, \tilde{f}_0 , we have

$$\begin{aligned} \tilde{e}_0 \cdot m_{T;j} &= \begin{cases} 0 & \text{if } i_1 \neq 1 \text{ or } i_\ell = n+1, \\ m_{(i_2, \dots, i_\ell, n+1); j-1} & \text{if } i_1 = 1 \text{ and } i_\ell \neq n+1, \end{cases} \\ \tilde{f}_0 \cdot m_{T;j} &= \begin{cases} 0 & \text{if } i_1 = 1 \text{ or } i_\ell \neq n+1, \\ m_{(1, i_1, \dots, i_{\ell-1}); j+1} & \text{if } i_1 \neq 1 \text{ and } i_\ell = n+1. \end{cases} \end{aligned}$$

By the above description, we have

Proposition 3.12. There is a bijection given by Ξ^0 between $\mathcal{M}_{I_0}(M_0)$ and $\mathcal{M}(V)$, where $V = V(\Xi^0(M_0))$ is the simple finite-dimensional $\mathcal{U}_q(\hat{\mathfrak{sl}}_{n+1})$ -module associated to $\Xi^0(M_0)$. In particular, the monomial crystal $\mathcal{M}_{I_0}(M_0)$ is I_0 -closed.

We determine in the next proposition when z_ℓ has the particular form of a shift.

Proposition 3.13. *There exists $p \in \mathbb{Z}$ such that the automorphism z_ℓ of $\mathcal{M}(Y_{\ell,0}Y_{0,\ell}^{-1})$ is equal to τ_p if and only if $\ell = 1$ or $\ell = r + 1$. Moreover, we have $z_1 = \tau_{-n-1}$ and $z_{r+1} = \tau_{-2}$.*

Proof. We have seen that $z_\ell = \sigma^{-1}$. So it suffices to determine when σ is a shift. We have the equality $\sigma^\ell = \tau_{n+1}$. Hence if $\ell = 1$, $\sigma = \tau_{n+1}$ is a shift. Assume that $\ell = r + 1$. In this case, $M_1 = \tau_2(M_0) = \sigma(M_0)$. As the crystal $\mathcal{M}(M_0)$ is connected and σ and τ_2 are automorphisms of crystals, we have $\sigma = \tau_2$. For the other cases, σ is explicitly known and is not a shift. \square

3.3. Affinized promotion operators and monomial crystals $\mathcal{M}(Y_{\ell,0}Y_{0,\ell}^{-1})$. In this section, we introduce promotion operators for the affinization of $\mathcal{U}_q(sl_{n+1})$ -crystals of finite type and for level 0 extremal fundamental weight crystals. We explicit them in the monomial realizations of $\mathcal{B}(\varpi_\ell)$ ($1 \leq \ell \leq n$).

Let us begin by some definitions and properties about the promotion operators (see [2, 16, 44, 45] and references therein for more details). In type A_n , the highest weight crystal $\mathcal{B}(\lambda)$ of highest weight λ can be realized by the semi-standard Young tableaux of shape (λ) with weight function wt given by the content of tableaux (i.e. $\text{wt}(b) := (w_1(b), \dots, w_{n+1}(b))$ where $w_i(b)$ is the number of letters i occurring in the tableau b).

Definition 3.14. *Let \mathcal{B} be a $\mathcal{U}_q(sl_{n+1})$ -crystal. A promotion operator pr on \mathcal{B} is an operator $\text{pr} : \mathcal{B} \rightarrow \mathcal{B}$ such that*

- (1) *pr shifts the content: if $\text{wt}(b) = (w_1, \dots, w_{n+1})$ is the content of $b \in \mathcal{B}$, then $\text{wt}(\text{pr}(b)) = (w_{n+1}, w_1, \dots, w_n)$,*
- (2) *promotion has order $n + 1$: $\text{pr}^{n+1} = \text{id}$,*
- (3) *$\text{pr} \circ \tilde{e}_i = \tilde{e}_{i+1} \circ \text{pr}$ and $\text{pr} \circ \tilde{f}_i = \tilde{f}_{i+1} \circ \text{pr}$ for $i \in \{1, 2, \dots, n - 1\}$.*

Given a promotion operator pr on a $\mathcal{U}_q(sl_{n+1})$ -crystal \mathcal{B} , one defines an associated affine P_{cl} -crystal by setting

$$\tilde{e}_0 := \text{pr}^{-1} \circ \tilde{e}_1 \circ \text{pr} \text{ and } \tilde{f}_0 := \text{pr}^{-1} \circ \tilde{f}_1 \circ \text{pr}.$$

It was shown in [45] that the $\mathcal{U}_q(sl_{n+1})$ -crystal $\mathcal{B}(\lambda)$ has an unique promotion operator pr when λ is rectangular (i.e. of the form $k\Lambda_\ell$ with $\ell \in I_0$ and $k \in \mathbb{N}^*$), given by the Schützenberger jeu-de-taquin process. Furthermore the affine P_{cl} -crystal obtained from $\mathcal{B}(k\Lambda_\ell)$ by using the promotion operator pr is isomorphic to the crystal basis of the irreducible finite-dimensional $\mathcal{U}_q(\hat{sl}_{n+1})'$ -module $W(k\Lambda_\ell)$ (see [30]).

From the affine P_{cl} -crystal $\mathcal{B}(k\Lambda_\ell)$, let us consider $\mathcal{B}(k\Lambda_\ell)_{\text{aff}}$ its affinization (see also [33]): this is the P -crystal $\mathbb{Q}[z, z^{-1}] \otimes \mathcal{B}(k\Lambda_\ell)$ such that for all $s \in \mathbb{Z}$ and $b \in \mathcal{B}(k\Lambda_\ell)$,

$$\text{wt}(z^s b) = \text{wt}(b) + s\delta, \quad \tilde{e}_i \cdot z^s b = z^{s+\delta_{i,0}}(\tilde{e}_i \cdot b), \quad \tilde{f}_i \cdot z^s b = z^{s-\delta_{i,0}}(\tilde{f}_i \cdot b).$$

We introduce the affinized promotion operator on $\mathcal{B}(k\Lambda_\ell)_{\text{aff}}$.

Definition 3.15. *Let us consider the crystal of finite type $\mathcal{B}(k\Lambda_\ell)$ ($k \in \mathbb{N}, \ell \in I_0$), pr its associated promotion operator and $\mathcal{B}(k\Lambda_\ell)_{\text{aff}}$ its affinization. The affinized promotion operator pr_{aff} on $\mathcal{B}(k\Lambda_\ell)_{\text{aff}}$ is the operator $\text{pr}_{\text{aff}} : \mathcal{B}(k\Lambda_\ell)_{\text{aff}} \rightarrow \mathcal{B}(k\Lambda_\ell)_{\text{aff}}$ such that for all $b \in \mathcal{B}(\Lambda_\ell)$ and $s \in \mathbb{Z}$,*

$$\text{pr}_{\text{aff}}(z^s b) = z^{s-w_{n+1}(b)} \text{pr}(b).$$

One checks easily the following statements.

Lemma 3.16. *The affinized promotion operator pr_{aff} of the crystal of finite type $\mathcal{B}(k\Lambda_\ell)$ shifts the content. It satisfies*

$$\text{pr}_{\text{aff}} \circ \tilde{e}_i = \tilde{e}_{i+1} \circ \text{pr}_{\text{aff}} \text{ and } \text{pr}_{\text{aff}} \circ \tilde{f}_i = \tilde{f}_{i+1} \circ \text{pr}_{\text{aff}}$$

for $i \in \{0, 1, \dots, n\}$ (where $\tilde{e}_{n+1}, \tilde{f}_{n+1}$ are understood to be \tilde{e}_0, \tilde{f}_0 respectively). It has infinite order, the weight of $\text{pr}_{\text{aff}}^{n+1}$ being $-k\ell\delta$.

As for all $\ell \in I_0$, the P -crystals $\mathcal{B}(\varpi_\ell)$ and $\mathcal{B}(\Lambda_\ell)_{\text{aff}}$ are isomorphic (see [33]), the affinized promotion operator $\text{pr}_{\text{aff}} : \mathcal{B}(\Lambda_\ell)_{\text{aff}} \rightarrow \mathcal{B}(\Lambda_\ell)_{\text{aff}}$ induces an operator on the level 0 fundamental extremal crystal $\mathcal{B}(\varpi_\ell)$. We call it promotion operator of $\mathcal{B}(\varpi_\ell)$, also denoted by pr_{aff} .

We want to explicit the promotion operators of $\mathcal{B}(\varpi_\ell)$ in the monomial realizations. For that, we recall the notion of twisted isomorphism of crystals (this definition appears in [2]).

Definition 3.17. *Let \mathcal{B} and \mathcal{B}' be crystals over two isomorphic Dynkin diagrams D and D' with vertices respectively indexed by I and I' and let $\theta : I \rightarrow I'$ be an isomorphism from D to D' . Then ϕ is a θ -twisted isomorphism if for all $b \in \mathcal{B}$ and $i \in I$,*

$$\tilde{f}_{\theta(i)} \cdot \phi(b) = \phi(\tilde{f}_i \cdot b) \text{ and } \tilde{e}_{\theta(i)} \cdot \phi(b) = \phi(\tilde{e}_i \cdot b).$$

Recall that one has defined an automorphism θ of the Dynkin diagram of type $A_n^{(1)}$ corresponding to a rotation such that $\theta(i) = i + 1$ ($i \in I$). Let $\phi : \mathcal{M} \rightarrow \mathcal{M}$ be a map defined by

$$\phi\left(\prod Y_{i,n}^{u_{i,n}}\right) = \prod Y_{i+1,n+1}^{u_{i,n}}.$$

Proposition 3.18. *The map $\phi : \mathcal{M} \rightarrow \mathcal{M}$ is a θ -twisted automorphism of the P -crystal \mathcal{M} . Furthermore it induces a θ -twisted automorphism of $\mathcal{M}(Y_{\ell,0}Y_{0,\ell}^{-1})$ for all $\ell \in I_0$.*

Proof. It is easy to check that the map ϕ preserves the crystal structure of \mathcal{M} . Let us show that ϕ induces a θ -twisted automorphism of $\mathcal{M}(Y_{\ell,0}Y_{0,\ell}^{-1})$ ($\ell \in I_0$). As $\mathcal{M}(Y_{\ell,0}Y_{0,\ell}^{-1})$ is connected and ϕ is a θ -twisted automorphism of \mathcal{M} , it suffices to show that $\phi(Y_{\ell,0}Y_{0,\ell}^{-1}) = Y_{\ell+1,1}Y_{1,\ell+1}^{-1}$ is in $\mathcal{M}(Y_{\ell,0}Y_{0,\ell}^{-1})$. But the monomial $Y_{\ell+1,1}Y_{1,\ell+1}^{-1}$ can be obtained by applying successively the Kashiwara operators $\tilde{f}_\ell, \tilde{f}_{\ell-1}, \dots, \tilde{f}_1$ on $Y_{\ell,0}Y_{0,\ell}^{-1}$ and hence is a vertex of $\mathcal{M}(Y_{\ell,0}Y_{0,\ell}^{-1})$. \square

Denote by $\varphi : \mathcal{B}(\Lambda_\ell)_{\text{aff}} \rightarrow \mathcal{M}(Y_{\ell,0}Y_{0,\ell}^{-1})$ the isomorphism of P -crystals between $\mathcal{B}(\Lambda_\ell)_{\text{aff}}$ and $\mathcal{M}(Y_{\ell,0}Y_{0,\ell}^{-1})$. It is explicitly given by

$$\varphi : z^s T \ (s \in \mathbb{Z}, T \in \mathcal{B}(\Lambda_\ell)) \mapsto m_{T,-s} \in \mathcal{M}(Y_{\ell,0}Y_{0,\ell}^{-1}).$$

The following result relates the θ -twisted automorphism ϕ of $\mathcal{M}(Y_{\ell,0}Y_{0,\ell}^{-1})$ to the promotion operator of $\mathcal{B}(\varpi_\ell)$ introduced above.

Proposition 3.19. *Let pr_{aff} be the promotion operator of $\mathcal{B}(\varpi_\ell)$ and $\phi : \mathcal{M}(Y_{\ell,0}Y_{0,\ell}^{-1}) \rightarrow \mathcal{M}(Y_{\ell,0}Y_{0,\ell}^{-1})$ the θ -twisted automorphism of $\mathcal{M}(Y_{\ell,0}Y_{0,\ell}^{-1})$ introduced above. The following diagram commutes*

$$\begin{array}{ccc} \mathcal{B}(\Lambda_\ell)_{\text{aff}} & \xrightarrow{\varphi} & \mathcal{M}(Y_{\ell,0}Y_{0,\ell}^{-1}) \\ \text{pr}_{\text{aff}} \downarrow & & \downarrow \phi \\ \mathcal{B}(\Lambda_\ell)_{\text{aff}} & \xrightarrow{\varphi} & \mathcal{M}(Y_{\ell,0}Y_{0,\ell}^{-1}) \end{array}$$

Proof. For $1 \leq k \leq n+1$ and $p \in \mathbb{Z}$, we have

$$\begin{aligned} \phi\left(\boxed{k}_p\right) &= \phi(Y_{k-1,p+k}^{-1}Y_{k,p+k-1}) = Y_{k,p+k+1}^{-1}Y_{k+1,p+k} \\ &= \begin{cases} \boxed{k+1}_p & \text{if } k \leq n, \\ \boxed{1}_{p+n+1} & \text{if } k = n+1. \end{cases} \end{aligned}$$

Fix $j \in \mathbb{Z}$ and $T = (1 \leq i_1 < i_2 < \dots < i_\ell \leq n+1)$ a Young tableau of shape Λ_ℓ . If $i_\ell \neq n+1$, we have

$$\begin{aligned} \phi(m_{T;j}) &= \phi\left(\prod_{p=1}^j \boxed{i_p}_{n-\ell-2p+2j+2} \times \prod_{p=j+1}^{\ell} \boxed{i_p}_{\ell+1-2p+2j}\right) \\ &= \prod_{p=1}^j \boxed{i_p+1}_{n-\ell-2p+2j+2} \times \prod_{p=j+1}^{\ell} \boxed{i_p+1}_{\ell+1-2p+2j} \\ &= m_{\text{pr}(T);j} = \varphi(\text{pr}_{\text{aff}}(z^{-j}T)). \end{aligned}$$

Assume that $i_\ell = n+1$. Then

$$\begin{aligned} \phi(m_{T;j}) &= \prod_{p=1}^j \boxed{i_p+1}_{n-\ell-2p+2j+2} \times \prod_{p=j+1}^{\ell-1} \boxed{i_p+1}_{\ell+1-2p+2j} \times \boxed{1}_{n-\ell+2j+2} \\ &= m_{\text{pr}(T);j+1} = \varphi(\text{pr}_{\text{aff}}(z^{-j}T)). \end{aligned}$$

□

3.4. Application of promotion operators to the study of $\mathcal{M}(Y_{\ell,0}Y_{0,\ell}^{-1})$. In this section, we use promotion operators of the monomial crystals $\mathcal{M}(Y_{\ell,0}Y_{0,\ell}^{-1})$ of $\mathcal{U}_q(\hat{\mathfrak{sl}}_{n+1})$ to obtain a new description of them, precisising results given in [27]. Moreover, we determine the $\ell \in I_0$ for which the crystals $\mathcal{M}(Y_{\ell,0}Y_{0,\ell}^{-1})$ are closed. We still assume that $n = 2r+1$ is odd and $\ell \in I_0$.

Let us begin with the following remarks. The monomials $\phi^j(Y_{\ell,0}Y_{0,\ell}^{-1}) = Y_{\ell+j,j}Y_{j,\ell+j}^{-1}$ will have a particular importance in the construction of level 0 extremal fundamental loop weight modules. One can explicit them thanks to the θ -twisted automorphism ϕ of \mathcal{M} in terms of Young tableaux

- if j is such that $\ell+j \leq n+1$, $Y_{\ell+j,j}Y_{j,\ell+j}^{-1} \in \mathcal{M}_{I_0}(M_0)$ and is equal to $m_{T;0}$ with $T = (j+1, j+2, \dots, j+\ell)$,

- if $1 \leq j \leq \ell - 1$, then $Y_{j,n-\ell+j+1}Y_{n-\ell+j+1,n+j+1}^{-1} \in \mathcal{M}_{I_0}(M_j)$ and is equal to $m_{T;j}$ with $T = (1, 2, \dots, j, n - \ell + j + 2, \dots, n + 1)$.

We will have to consider the finite sub- I_j -crystals of $\mathcal{M}(Y_{\ell,0}Y_{0,\ell}^{-1})$

$$\mathcal{M}_{I_j}(Y_{\ell+j,j+k(n+1)}Y_{j,\ell+j+k(n+1)}^{-1})$$

for $j \in I$ and $k \in \mathbb{Z}$. Note that one of these crystals can be obtained from an other one by application of powers of ϕ .

Proposition 3.20. *Let $\ell \in I_0$. We have the equality of sets*

$$\mathcal{M}(Y_{\ell,0}Y_{0,\ell}^{-1}) = \bigcup_{k \in \mathbb{Z}} \tau_{n+1}^k \left(\bigcup_{j=0}^n \mathcal{M}_{I_j}(Y_{\ell+j,j}Y_{j,\ell+j}^{-1}) \right).$$

Proof. As $Y_{\ell+j,j}Y_{j,\ell+j}^{-1} \in \mathcal{M}(Y_{\ell,0}Y_{0,\ell}^{-1})$ for all $0 \leq j \leq n$ and $\mathcal{M}(Y_{\ell,0}Y_{0,\ell}^{-1})$ is connected,

$$\bigcup_{j=0}^n \mathcal{M}_{I_j}(Y_{\ell+j,j}Y_{j,\ell+j}^{-1}) \subset \mathcal{M}(Y_{\ell,0}Y_{0,\ell}^{-1})$$

as sets.

Let us fix $m_{T;j} \in \mathcal{M}(Y_{\ell,0}Y_{0,\ell}^{-1})$ with $0 \leq j \leq \ell - 1$ and $T = (1 \leq i_1 < i_2 < \dots < i_\ell \leq n + 1)$. We will show that $m_{T;j} \in \bigcup_{j=0}^n \mathcal{M}_{I_j}(Y_{\ell+j,j}Y_{j,\ell+j}^{-1})$. If $j = 0$, we have $m_{T;0} \in \mathcal{M}_{I_0}(Y_{\ell,0}Y_{0,\ell}^{-1})$. Assume that $1 \leq j \leq \ell - 1$ and set $s = i_{j+1} - 1$. Then $T = (i_1 < \dots < i_j < s + 1 < i_{j+2} < \dots < i_\ell)$ and by application of the Kashiwara operators $\tilde{e}_1, \dots, \tilde{e}_{s-1}, \tilde{e}_{s+2}, \dots, \tilde{e}_n$ on $m_{T;j}$, we show that

$$m_{T;j} \in \mathcal{M}_{I_s}(m_{T';j}) \text{ with } T' = (1 < \dots < j < s + 1 < \dots < s + \ell - j).$$

By applying $\tilde{e}_1, \dots, \tilde{e}_{j-1}, \tilde{e}_{s+\ell-j+1}, \dots, \tilde{e}_n$ and \tilde{e}_0 on $m_{T';j}$, it is sent on

$$m_{T'';0} \text{ with } T'' = (s + 1 < \dots < s + \ell) \text{ if } s + \ell \leq n + 1,$$

and on

$$m_{T'';u} \text{ with } u = s + \ell - n - 1, T'' = (1 < \dots < u < s + 1 < \dots < n + 1) \text{ otherwise.}$$

Furthermore $m_{T'';u} = \phi^s(Y_{\ell,0}Y_{0,\ell}^{-1}) = Y_{\ell+s,s}Y_{s,\ell+s}^{-1}$ by the above remark and $m_{T;j}$ is also contained in $\mathcal{M}_{I_s}(Y_{\ell+s,s}Y_{s,\ell+s}^{-1})$. \square

Remark 3.21. *One of the questions treated in [27, Section 4] is to give an explicit description of monomials occurring in $\mathcal{M}(Y_{\ell,0}Y_{0,\ell}^{-1})$. The preceding proposition gives*

precisions in the following way: the $\ell \cdot \binom{n+1}{\ell}$ monomials in $\mathcal{M}(Y_{\ell,0}Y_{0,\ell}^{-1})$ can be obtained from the $I_{\{0,1\}}$ -crystal $\mathcal{M}_{I_{\{0,1\}}}(Y_{\ell,0}Y_{0,\ell}^{-1})$ with only $\binom{n}{\ell-1}$ vertices by applying ϕ . In fact, a monomial $m_{T';0} \in \mathcal{M}_{I_{\{0,1\}}}(Y_{\ell,0}Y_{0,\ell}^{-1})$ is such that T' has the form $T' = (1 < i_2 < \dots < i_\ell)$. And all the others $m_{T;j}$ can be obtained from them by applying powers of ϕ .

We determine when the monomial crystal $\mathcal{M}(Y_{\ell,0}Y_{0,\ell}^{-1})$ is closed. For that, we need the following lemma which will be also used in the next section.

Lemma 3.22. *Assume that z_ℓ is a shift τ_{-p} . For any $0 \leq j \leq n$, we have the equality of I_j -crystals*

$$(13) \quad \mathcal{M}(Y_{\ell,0}Y_{0,\ell}^{-1}) = \bigsqcup_{k \in \mathbb{Z}} \tau_p^k \left(\mathcal{M}_{I_j}(Y_{\ell+j,j}Y_{j,\ell+j}^{-1}) \right)$$

Proof. This is straightforward from (12) and the fact that τ_{-p} and ϕ commute. \square

Proposition 3.23. *The monomial crystal $\mathcal{M}(Y_{\ell,0}Y_{0,\ell}^{-1})$ is closed if and only if $\ell = 1$ or $\ell = r + 1$. This corresponds to the case when z_ℓ has the form of a shift.*

Proof. Assume that $2 \leq \ell \leq r$ and consider the monomial

$$M_j = Y_{\ell,2j}Y_{0,n-\ell+1+2j}^{-1}Y_{j,\ell+j}^{-1}Y_{j,n-\ell+1+j} \in \mathcal{M}(Y_{\ell,0}Y_{0,\ell}^{-1})$$

with $j \neq 0$. If the crystal $\mathcal{M}(Y_{\ell,0}Y_{0,\ell}^{-1})$ is q -closed, the monomial

$$m = Y_{\ell,2j}Y_{0,n-\ell+1+2j}^{-1}Y_{j,\ell+j-2}Y_{j-1,\ell+j-1}^{-1}Y_{j+1,\ell+j-1}^{-1}Y_{j,n-\ell+1+j}$$

has to be a vertex of it. But it is not the case, m being not of the form (11).

Now assume that $\ell = 1$ or $r + 1$. In this case, $z_\ell = \tau_{-p}$ with $p = n + 1$ or $p = 2$ respectively and by the above lemma

$$\mathcal{M}(Y_{\ell,0}Y_{0,\ell}^{-1}) = \bigsqcup_{k \in \mathbb{Z}} \tau_p^k \left(\mathcal{M}_{I_j}(Y_{\ell+j,j}Y_{j,\ell+j}^{-1}) \right)$$

as I_j -crystals ($0 \leq j \leq n$). As the finite crystal $\mathcal{M}_{I_0}(M_0)$ is I_0 -closed, the I_j -crystals $\mathcal{M}_{I_j}(Y_{\ell+j,j}Y_{j,\ell+j}^{-1})$ are I_j -closed for all $0 \leq j \leq n$ and $\mathcal{M}(Y_{\ell,0}Y_{0,\ell}^{-1})$ is closed by the above equalities. \square

4. LEVEL 0 EXTREMAL FUNDAMENTAL LOOP WEIGHT MODULES FOR $\mathcal{U}_q(sl_{n+1}^{tor})$ WHEN $\mathcal{M}(Y_{\ell,0}Y_{0,\ell}^{-1})$ IS CLOSED

Assume that $n = 2r + 1$ is odd ($r \geq 1$) and $\mathcal{M}(Y_{\ell,0}Y_{0,\ell}^{-1})$ is closed (it holds if and only if $\ell = 1$ or $\ell = r + 1$). In this section, we relate the monomial crystals $\mathcal{M}(Y_{\ell,0}Y_{0,\ell}^{-1})$ ($\ell \in I_0$) of $\mathcal{U}_q(\hat{sl}_{n+1})$ with integrable representations of $\mathcal{U}_q(sl_{n+1}^{tor})$, which we expect to be extremal loop weight modules.

In the first part, we construct a new infinite family of representations $V(Y_{\ell,0}Y_{0,\ell}^{-1})$ of $\mathcal{U}_q(sl_{n+1}^{tor})$ (Theorem 4.1). We call these representations the level 0 extremal fundamental loop weight modules. Let us give the outline of this construction: consider the vector space $V(Y_{\ell,0}Y_{0,\ell}^{-1})$ freely generated by the monomials occurring in $\mathcal{M}(Y_{\ell,0}Y_{0,\ell}^{-1})$. For all $0 \leq j \leq n$, we define a structure of $\mathcal{U}_q^{v,j}(sl_{n+1}^{tor})$ -module on it denoted by $V(Y_{\ell,0}Y_{0,\ell}^{-1})^{(j)}$ such that

$$V(Y_{\ell,0}Y_{0,\ell}^{-1})^{(j)} = \bigoplus_{k \in \mathbb{Z}} V_k^{(j)}$$

where $V_k^{(j)}$ is generated by the $m \in \mathcal{M}_{I_j}(Y_{\ell+j,j+k}Y_{j,\ell+j+k}^{-1})$ as vector space, endowed with a structure of $\mathcal{U}_q^{v,j}(sl_{n+1}^{tor})$ -module isomorphic to $V(Y_{\ell,j+k}Y_{0,\ell+j+k}^{-1})^{(j)}$. This decomposition can be compared to the equalities of crystals (13). We define in this way an action of $\mathcal{U}_q(sl_{n+1}^{tor})$ on $V(Y_{\ell,0}Y_{0,\ell}^{-1})$, the compatibility between the action of various vertical subalgebras being a consequence of the existence of promotion operators on $\mathcal{M}(Y_{\ell,0}Y_{0,\ell}^{-1})$. Furthermore the q -character of $V(Y_{\ell,0}Y_{0,\ell}^{-1})$ is the sum of monomials occurring in $\mathcal{M}(Y_{\ell,0}Y_{0,\ell}^{-1})$ with multiplicity one.

In the second part, we study these representations: we show that $V(Y_{\ell,0}Y_{0,\ell}^{-1})$ is irreducible and it is an extremal loop weight modules, generated by an extremal vector $v_{Y_{\ell,0}Y_{0,\ell}^{-1}}$ of ℓ -weight $Y_{\ell,0}Y_{0,\ell}^{-1}$. Furthermore explicit formulas are given for the action of $\mathcal{U}_q(sl_{n+1}^{tor})$ on $V(Y_{\ell,0}Y_{0,\ell}^{-1})$. It is remarkable that these formulas are expressed only in terms of the constants of the associated monomial crystal and are “universal” in the following sense: the action on all the level 0 extremal fundamental loop weight modules $V(Y_{\ell,0}Y_{0,\ell}^{-1})$ is completely determined by these formulas and by the data of the corresponding monomial crystals $\mathcal{M}(Y_{\ell,0}Y_{0,\ell}^{-1})$. This sheds new light on the link between monomial crystals and the theory of q -characters already expected in [27]. All these sentences hold for the fundamental ℓ -highest weight modules $V(Y_{\ell,0})$ of $\mathcal{U}_q(\hat{sl}_{n+1})$ with the corresponding monomial crystals $\mathcal{M}(Y_{\ell,0})$.

In the third part, we specialize q at a root of unity ϵ . We obtain new irreducible finite-dimensional representations of the quantum toroidal algebra $\mathcal{U}_\epsilon(sl_{n+1}^{tor})$.

4.1. Construction of the level 0 extremal fundamental loop weight modules.

Let us begin by the main result of this section.

Theorem 4.1. *Assume that $n = 2r + 1$ is odd and $\ell = 1$ or $r + 1$. There exists a representation of $\mathcal{U}_q(sl_{n+1}^{tor})$ whose q -character is the sum of all monomials occurring in $\mathcal{M}(Y_{\ell,0}Y_{0,\ell}^{-1})$ with multiplicity one. It is denoted by $V(Y_{\ell,0}Y_{0,\ell}^{-1})$ and called the level 0 extremal fundamental loop weight module of extremal ℓ -weight $Y_{\ell,0}Y_{0,\ell}^{-1}$.*

To construct these representations, let us start with results about the fundamental modules $V(Y_{\ell,k})$ of $\mathcal{U}_q(\hat{sl}_{n+1})$ ($n \in \mathbb{N}^*$, $1 \leq \ell \leq n$, $k \in \mathbb{Z}$). As a $\mathcal{U}_q(sl_{n+1})$ -module, it is isomorphic to the fundamental highest weight module $V(\Lambda_\ell)$. So we begin by recalling some well-known facts about $V(\Lambda_\ell)$ which will be useful.

Lemma 4.2. *All the weight spaces of the fundamental highest weight module $V(\Lambda_\ell)$ ($1 \leq \ell \leq n$) are of dimension 1. Furthermore the Weyl group of finite type W acts transitively on $\text{wt}(V(\Lambda_\ell))$.*

Proof. We give here a proof of these results by considering the associated crystal $\mathcal{B}(\Lambda_\ell)$ of semi-standard Young tableaux $T = (1 \leq i_1 < \dots < i_\ell \leq n+1)$ of shape (Λ_ℓ) .

For all Young tableau T of shape $\lambda \in \mathbb{N}\Lambda_1 \oplus \dots \oplus \mathbb{N}\Lambda_n$, we consider $\text{wt}(T) = (w_1, w_2, \dots, w_{n+1})$ as an element of P as follows. Set $\epsilon_i = \Lambda_i - \Lambda_{i-1}$ for $2 \leq i \leq n$, $\epsilon_1 = \Lambda_1$ and $\epsilon_{n+1} = -\epsilon_1 - \dots - \epsilon_n$. In particular, $\alpha_i = \epsilon_i - \epsilon_{i+1}$ and we have $P = \mathbb{Z}\epsilon_1 \oplus \dots \oplus \mathbb{Z}\epsilon_{n+1}$. Set $\text{wt}(T) = w_1\epsilon_1 + \dots + w_{n+1}\epsilon_{n+1}$. For $T = (1 \leq i_1 < \dots < i_\ell \leq n+1)$ of shape (Λ_ℓ) , we have $\text{wt}(T) = \epsilon_{i_1} + \dots + \epsilon_{i_{n+1}}$. These weights are different to each

other and for all $\nu \in \text{wt}(V(\Lambda_\ell))$,

$$\dim(V(\Lambda_\ell)_\nu) = \sharp \mathcal{B}(\Lambda_\ell)_\nu = 1.$$

Furthermore $\Lambda_\ell = \epsilon_1 + \dots + \epsilon_\ell$ and the action of the simple reflections s_i on ϵ_j ($1 \leq i \leq n$, $1 \leq j \leq n+1$) is

$$s_i(\epsilon_j) = \begin{cases} \epsilon_{i+1} & \text{if } i = j, \\ \epsilon_i & \text{if } i = j+1, \\ \epsilon_j & \text{if } i \neq j, j+1. \end{cases}$$

Hence, W acts transitively on $\text{wt}(V(\Lambda_\ell))$. \square

Proposition 4.3. *Let $V(Y_{\ell,k})$ be a fundamental module of $\mathcal{U}_q(\hat{sl}_{n+1})$ ($\ell \in I_0, k \in \mathbb{Z}$). There exists a basis (v_m) of $V(Y_{\ell,k})$ indexed by the vertices of the monomial crystal $\mathcal{M}(Y_{\ell,k})$ such that for all $i \in I_0$ and $m \in \mathcal{M}(Y_{\ell,k})$,*

$$\text{wt}(v_m) = m, \quad x_{i,0}^+ \cdot v_m = v_{\tilde{e}_i \cdot m}, \quad x_{i,0}^- \cdot v_m = v_{\tilde{f}_i \cdot m},$$

where $v_0 = 0$ by convention.

Proof. As a $\mathcal{U}_q(\hat{sl}_{n+1})$ -module, $V(Y_{\ell,k})$ is the extremal weight module of dominant weight Λ_ℓ , generated by an extremal vector of weight Λ_ℓ , denoted by v . Hence, there exists $\{v_w\}_{w \in W}$ such that $v_{\text{Id}} = v$ and

$$x_{i,0}^\pm \cdot v_w = 0 \text{ if } \pm w(\lambda)(h_i) \geq 0 \text{ and } (x_{i,0}^\mp)^{(\pm w(\lambda)(h_i))} \cdot v_w = v_{s_i(w)}.$$

By the above lemma and the fact that v_w is of weight $w(\Lambda_\ell)$ for all $w \in W$, $\{v_w\}_{w \in W}$ is a basis of $V(Y_{\ell,k})$.

We reindex this basis in the following way. The weight subspaces and the ℓ -weight subspaces of $V(Y_{\ell,k})$ coincide and are of dimension one. As consequences, all the v_w are ℓ -weight vectors and for all $m \in \mathcal{M}(V(Y_{\ell,k})) = \mathcal{M}(Y_{\ell,k})$ there is a unique $w \in W$ such that v_w is of ℓ -weight m . So for all $m \in \mathcal{M}(Y_{\ell,k})$, let us define v_m as the unique vector v_w of ℓ -weight m .

We determine the action of $\mathcal{U}_q^h(\hat{sl}_{n+1})$ on this basis. Fix $m \in \mathcal{M}(Y_{\ell,k})$ and let $w \in W$ be such that $v_m = v_w$ (w is determined by the equality $w(\Lambda_\ell) = \text{wt}(m)$). For all $i \in I_0$, we have $\text{wt}(m)(h_i) = -1, 0, 1$. Then $x_{i,0}^\pm \cdot v_w$ is equal to zero or is of the form $v_{s_i(w)}$. In the first case, there is no monomial in $\mathcal{M}(V(Y_{\ell,k})) = \mathcal{M}(Y_{\ell,k})$ of weight $\text{wt}(m) \pm \alpha_i$ and $\tilde{e}_i \cdot m = \tilde{f}_i \cdot m = 0$. In the other case, there is a unique monomial in $\mathcal{M}(V(Y_{\ell,k})) = \mathcal{M}(Y_{\ell,k})$ of weight $\text{wt}(m) \pm \alpha_i$: it is respectively $\tilde{e}_i \cdot m$ or $\tilde{f}_i \cdot m$. Hence we have $x_{i,0}^+ \cdot v_m = v_{\tilde{e}_i \cdot m}$ and $x_{i,0}^- \cdot v_m = v_{\tilde{f}_i \cdot m}$. \square

In particular, the action of $\mathcal{U}_q(\hat{sl}_{n+1})$ on the fundamental modules $V(Y_{\ell,k})$ is determined by the combinatorics of monomial crystals $\mathcal{M}(Y_{\ell,k})$: in fact, the action of operators $x_{i,r}^\pm$ ($1 \leq i \leq n$, $r \in \mathbb{Z}$) deduces from the action of the $x_{i,0}^\pm$ (given by $\mathcal{M}(Y_{\ell,k})$) and the action of $h_{i,r}$ (given by the ℓ -weights $m \in \mathcal{M}(Y_{\ell,k})$) from (7).

Let us begin the construction of level 0 extremal fundamental loop weight modules. Assume that $n = 2r+1$ is odd and consider the monomial crystal $\mathcal{M}(Y_{\ell,0}Y_{0,\ell}^{-1})$, supposed

to be closed. This is the case if and only if $\ell = 1$ or $\ell = r + 1$ (Proposition 3.23). Set $p = n + 1$ or $p = 2$ respectively.

Denote by \mathcal{E} (resp. $\mathcal{E}_{j,k}$ for $0 \leq j \leq n$ and $k \in \mathbb{Z}$) the set of monomials occurring in $\mathcal{M}(Y_{\ell,0}Y_{0,\ell}^{-1})$ (resp. in $\tau_p^k(\mathcal{M}_{I_j}(Y_{\ell+j,j}Y_{j,\ell+j}^{-1})) = \mathcal{M}_{I_j}(Y_{\ell+j,j+kp}Y_{j,\ell+j+kp}^{-1})$). By (13), one has $\mathcal{E} = \bigsqcup_{k \in \mathbb{Z}} \mathcal{E}_{j,k}$ for all $0 \leq j \leq n$.

Let

$$(14) \quad V(Y_{\ell,0}Y_{0,\ell}^{-1}) = \bigoplus_{m \in \mathcal{E}} \mathbb{C}v_m$$

be the vector space freely generated by elements of \mathcal{E} . For all $0 \leq j \leq n$ and $k \in \mathbb{Z}$ set $V_k^{(j)} = \bigoplus_{m \in \mathcal{E}_{j,k}} \mathbb{C}v_m$ the subspace of V of dimension $\dim(V(\Lambda_\ell))$. In particular, we have

$$V(Y_{\ell,0}Y_{0,\ell}^{-1}) = \bigoplus_{k \in \mathbb{Z}} V_k^{(j)}.$$

We endow the vector space $V(Y_{\ell,0}Y_{0,\ell}^{-1})$ with a structure of $\mathcal{U}_q^{v,j}(sl_{n+1}^{tor})$ -module as follows ($0 \leq j \leq n$): for all $k \in \mathbb{Z}$, let (v_m) be the basis of the $\mathcal{U}_q^{v,j}(sl_{n+1}^{tor})$ -module $V(Y_{\ell,j+kp})^{(j)}$ defined in Proposition 4.3 indexed by the I_j -crystal $\mathcal{M}_{I_j}(Y_{\ell+j,j+kp})$. Let us define an isomorphism of vector spaces between $V_k^{(j)}$ and $V(Y_{\ell,j+kp})^{(j)}$ by

$$\begin{aligned} V_k^{(j)} &\longrightarrow V(Y_{\ell,j+kp})^{(j)} \\ v_m &\mapsto v_{\Xi^j(m)} \end{aligned}$$

We endow the vector space $V_k^{(j)}$ with a structure of $\mathcal{U}_q^{v,j}(sl_{n+1}^{tor})$ -module by pull back the action of $\mathcal{U}_q^{v,j}(sl_{n+1}^{tor})$ on $V(Y_{\ell,j+kp})^{(j)}$. By direct sum, $V(Y_{\ell,0}Y_{0,\ell}^{-1})$ is a $\mathcal{U}_q^{v,j}(sl_{n+1}^{tor})$ -module, denoted by $V(Y_{\ell,0}Y_{0,\ell}^{-1})^{(j)}$.

Proposition 4.4. *There exists a structure of $\mathcal{U}_q(sl_{n+1}^{tor})$ -module on $V(Y_{\ell,0}Y_{0,\ell}^{-1})$ such that for all $j \in I$ the induced $\mathcal{U}_q^{v,j}(sl_{n+1}^{tor})$ -module is isomorphic to $V(Y_{\ell,0}Y_{0,\ell}^{-1})^{(j)}$. Furthermore the q -character of $V(Y_{\ell,0}Y_{0,\ell}^{-1})$ is*

$$\chi_q(V(Y_{\ell,0}Y_{0,\ell}^{-1})) = \sum_{m \in \mathcal{E}} m,$$

where \mathcal{E} is the set of the monomials occurring in $\mathcal{M}(Y_{\ell,0}Y_{0,\ell}^{-1})$.

Proof. To define an action of $\mathcal{U}_q(sl_{n+1}^{tor})$ on $V(Y_{\ell,0}Y_{0,\ell}^{-1})$, it suffices to determine the action of the subalgebras $\hat{\mathcal{U}}_i$ for all $i \in I$. For that, let $j \in I$ be such that $j \neq i$. The action of $\hat{\mathcal{U}}_i$ on $V(Y_{\ell,0}Y_{0,\ell}^{-1})$ is the restriction of the action of $\mathcal{U}_q^{v,j}(sl_{n+1}^{tor})$ on $V(Y_{\ell,0}Y_{0,\ell}^{-1})^{(j)}$.

This definition is independent of the choice of $j \in I$, $j \neq i$: for $m \in \mathcal{E}$, the action of $\mathcal{U}_q^{v,j}(sl_{n+1}^{tor})$ on the vector v_m is determined by the sub- I_j -crystal $\mathcal{M}_{I_j}(m)$ of $\mathcal{M}(Y_{\ell,0}Y_{0,\ell}^{-1})$ and by the ℓ -weight $\Xi^j(m)$. So the action of $\hat{\mathcal{U}}_i$ on v_m is determined by the action of \tilde{e}_i and \tilde{f}_i on m and by the ℓ -weight $\Xi_i(m)$ which are independent of the choice of j .

Let us show that this action endows $V(Y_{\ell,0}Y_{0,\ell}^{-1})$ with a structure of $\mathcal{U}_q(sl_{n+1}^{tor})$ -module. We fix two indices $i_1, i_2 \in I$ and we check the relations satisfied by $\hat{\mathcal{U}}_{i_1}$ and $\hat{\mathcal{U}}_{i_2}$. The

indices i_1 and i_2 are in the same connected subset I_j of the set of vertices of the Dynkin diagram ($j \in I$). By construction, the action of $\hat{\mathcal{U}}_{i_1}$ and $\hat{\mathcal{U}}_{i_2}$ are restrictions of the action of $\mathcal{U}_q^{v,j}(sl_{n+1}^{tor})$ on $V(Y_{\ell,0}Y_{0,\ell}^{-1})$. As $V(Y_{\ell,0}Y_{0,\ell}^{-1})^{(j)}$ is a $\mathcal{U}_q^{v,j}(sl_{n+1}^{tor})$ -module, the relations between $\hat{\mathcal{U}}_{i_1}$ and $\hat{\mathcal{U}}_{i_2}$ are satisfied and $V(Y_{\ell,0}Y_{0,\ell}^{-1})$ is a $\mathcal{U}_q(sl_{n+1}^{tor})$ -module.

By construction, the induced $\mathcal{U}_q^{v,j}(sl_{n+1}^{tor})$ -module of $V(Y_{\ell,0}Y_{0,\ell}^{-1})$ is isomorphic to $V(Y_{\ell,0}Y_{0,\ell}^{-1})^{(j)}$ for all $j \in I$. Furthermore the ℓ -weight of v_m is $\Xi_i(m)$ for the action of $\hat{\mathcal{U}}_i$ ($i \in I$). So m is the ℓ -weight of v_m and the q -character of $V(Y_{\ell,0}Y_{0,\ell}^{-1})$ is the sum of monomials occurring in $\mathcal{M}(Y_{\ell,0}Y_{0,\ell}^{-1})$ with multiplicity one. \square

4.2. Study of the level 0 extremal fundamental loop weight modules. In this section, we study the $\mathcal{U}_q(sl_{n+1}^{tor})$ -modules $V(Y_{\ell,0}Y_{0,\ell}^{-1})$, where $n = 2r + 1$ is supposed to be odd and $\ell = 1$ or $\ell = r + 1$. We set $p = n + 1$ or $p = 2$ respectively.

Proposition 4.5. *The $\mathcal{U}_q(sl_{n+1}^{tor})$ -module $V(Y_{\ell,0}Y_{0,\ell}^{-1})$ is integrable. Moreover, it satisfies properties (3) and (4) of Remark 2.4 with weight subspaces of dimension one.*

Proof. The q -character of $V(Y_{\ell,0}Y_{0,\ell}^{-1})$ is known: this is the sum of monomials occurring in $\mathcal{M}(Y_{\ell,0}Y_{0,\ell}^{-1})$ with multiplicity one. Furthermore one has the equality of I_0 -crystals

$$\mathcal{M}(Y_{\ell,0}Y_{0,\ell}^{-1}) = \bigsqcup_{k \in \mathbb{Z}} \tau_p^k \left(\mathcal{M}_{I_0}(Y_{\ell,0}Y_{0,\ell}^{-1}) \right).$$

For all $m \in \mathcal{M}_{I_0}(Y_{\ell,0}Y_{0,\ell}^{-1})$ and $k \in \mathbb{Z}$, $\text{wt}(\tau_p^k(m)) = \text{wt}(m) - k\delta$. So to prove that the weight spaces of $V(Y_{\ell,0}Y_{0,\ell}^{-1})$ are of dimension one, we have to show that the weights of monomials occurring in $\mathcal{M}_{I_0}(Y_{\ell,0}Y_{0,\ell}^{-1})$ are different to each other. More precisely, it is sufficient to show that the sum

$$\sum_{m \in \mathcal{M}_{I_0}(Y_{\ell,0}Y_{0,\ell}^{-1})} \text{wt}(\Xi^0(m)) \in \mathbb{Z}\Lambda_1 \oplus \cdots \oplus \mathbb{Z}\Lambda_n$$

is without multiplicity. This is a consequence of the above results: this is the character of the $\mathcal{U}_q(sl_{n+1})$ -module $V(\Lambda_\ell)$.

For all $j \in I$, the representation $V(Y_{\ell,0}Y_{0,\ell}^{-1})$ is completely reducible as a $\mathcal{U}_q^{v,j}(sl_{n+1}^{tor})$ -module and we have

$$(15) \quad V(Y_{\ell,0}Y_{0,\ell}^{-1})^{(j)} = \bigoplus_{p \in \mathbb{Z}} V(Y_{\ell,j+kp})^{(j)}.$$

As the representations $V(Y_{\ell,j+kp})$ are all integrable, it holds for $V(Y_{\ell,0}Y_{0,\ell}^{-1})$. Furthermore $V(Y_{\ell,0}Y_{0,\ell}^{-1})$ satisfies the stronger property (4) of Remark 2.4: in fact the representations $V(Y_{\ell,j+kp})$ are all isomorphic as $\mathcal{U}_q(sl_{n+1})$ -modules and satisfy property (4). Hence we have $V(Y_{\ell,0}Y_{0,\ell}^{-1})_{\nu+N\alpha_i} = \{0\}$ for all $\nu \in P$, $i \in I$, $N \gg 0$. \square

Theorem 4.6. *The $\mathcal{U}_q(sl_{n+1}^{tor})$ -module $V(Y_{\ell,0}Y_{0,\ell}^{-1})$ is an extremal loop weight module generated by the vector $v_{Y_{\ell,0}Y_{0,\ell}^{-1}}$ of ℓ -weight $Y_{\ell,0}Y_{0,\ell}^{-1}$.*

Proof. The formulas (15) imply immediately the third point of Definition 2.21. The first two points are consequences of the following lemmas. \square

Lemma 4.7. *Let \mathcal{M}' be a sub- $\mathcal{U}_q(\hat{sl}_{n+1})$ -crystal of \mathcal{M} . Assume that V is a $\mathcal{U}_q(\hat{sl}_{n+1})$ -module with basis $(v_m)_{m \in \mathcal{M}'}$ satisfying*

$$(16) \quad \text{wt}(v_m) = \text{wt}(m), \quad x_i^+ \cdot v_m = v_{\tilde{e}_i \cdot m} \quad \text{and} \quad x_i^- \cdot v_m = v_{\tilde{f}_i \cdot m}$$

for all $m \in \mathcal{M}'$ and $i \in I$, where $v_0 = 0$ by convention. If the monomial m is extremal of weight λ , then the vector v_m is an extremal vector of V of weight λ . Furthermore if the crystal \mathcal{M}' is connected, then the $\mathcal{U}_q(\hat{sl}_{n+1})$ -module V is cyclic generated by any v_m with $m \in \mathcal{M}'$.

Proof. Assume that m is extremal of weight λ : there exists $\{m_w\}_{w \in \hat{W}}$ such that $m_{Id} = m$ and

$$(17) \quad \begin{aligned} \tilde{e}_i \cdot m_w &= 0 \text{ if } w(\lambda)(h_i) \geq 0 \text{ and } (\tilde{f}_i)^{w(\lambda)(h_i)} \cdot m_w = m_{s_i(w)}, \\ \tilde{f}_i \cdot m_w &= 0 \text{ if } w(\lambda)(h_i) \leq 0 \text{ and } (\tilde{e}_i)^{-w(\lambda)(h_i)} \cdot m_w = m_{s_i(w)}. \end{aligned}$$

For all $w \in \hat{W}$, set $v_w = v_{m_w}$. By (16) and (17), $\{v_w\}_{w \in \hat{W}}$ satisfies $v_{Id} = v_m$ and

$$x_i^\pm \cdot v_w = 0 \text{ if } \pm w(\lambda)(h_i) \geq 0 \text{ and } (x_i^\mp)^{(\pm w(\lambda)(h_i))} \cdot v_w = v_{s_i(w)}.$$

Hence the vector v_m is extremal of weight λ .

Assume that the crystal \mathcal{M}' is connected and fix $m \in \mathcal{M}'$. For $m' \in \mathcal{M}'$, there exists a product s of Kashiwara operators such that $s(m) = m'$. Consider the corresponding operator $S \in \mathcal{U}_q(\hat{sl}_{n+1})$ at the level of V , i.e. S has the same expression as s where the operators \tilde{e}_i (resp. \tilde{f}_i) are replaced by x_i^+ (resp. x_i^-) in the product. By (16), $S(v_m) = v_{s(m)} = v_{m'}$ and the $\mathcal{U}_q(\hat{sl}_{n+1})$ -module V is cyclic generated by v_m . \square

Lemma 4.8. *The ℓ -weight vector $v_{Y_{\ell,0}Y_{0,\ell}^{-1}} \in V(Y_{\ell,0}Y_{0,\ell}^{-1})$ is an extremal vector of weight ϖ_ℓ for the action of $\mathcal{U}_q^h(sl_{n+1}^{tor})$. Furthermore*

$$V(Y_{\ell,0}Y_{0,\ell}^{-1}) = \mathcal{U}_q^h(sl_{n+1}^{tor}) \cdot v_{Y_{\ell,0}Y_{0,\ell}^{-1}}.$$

Proof. Let us begin to show that the basis (v_m) of $V(Y_{\ell,0}Y_{0,\ell}^{-1})$ introduced in (14) satisfies properties (16). For all $m \in \mathcal{M}(Y_{\ell,0}Y_{0,\ell}^{-1})$, v_m is an ℓ -weight vector of ℓ -weight m and $\text{wt}(v_m) = \text{wt}(m)$. Fix $i \in I$ and let $j \in I$ be such that $j \neq i$. As $\mathcal{U}_q^{v,j}(sl_{n+1}^{tor})$ -module, $V(Y_{\ell,0}Y_{0,\ell}^{-1})$ is completely reducible (see (15)) and there exists $k \in \mathbb{Z}$ such that $v_m \in V(Y_{\ell,j+kp})^{(j)}$. As properties (16) are satisfied in $V(Y_{\ell,j+kp})^{(j)}$ by Proposition 4.3, it holds on v_m for $i \in I$.

From there the result is a direct consequence of the fact that the monomial $Y_{\ell,0}Y_{0,\ell}^{-1}$ is an extremal element of the connected crystal $\mathcal{M}(Y_{\ell,0}Y_{0,\ell}^{-1})$ (Theorem 3.5). \square

Proposition 4.9. *The $\mathcal{U}_q(sl_{n+1}^{tor})$ -module $V(Y_{\ell,0}Y_{0,\ell}^{-1})$ is irreducible. Furthermore it is simple as a $\mathcal{U}_q^h(sl_{n+1}^{tor})$ -module, isomorphic to $V(\varpi_\ell)$.*

Proof. Let W be a non trivial sub- $\mathcal{U}_q^h(sl_{n+1}^{tor})$ -module of $V(Y_{\ell,0}Y_{0,\ell}^{-1})$. As the weight spaces of $V(Y_{\ell,0}Y_{0,\ell}^{-1})$ are all of dimension one, there exists $m \in \mathcal{M}(Y_{\ell,0}Y_{0,\ell}^{-1})$ such that $v_m \in W$. By Lemma 4.7, v_m generates $V(Y_{\ell,0}Y_{0,\ell}^{-1})$ and $W = V(Y_{\ell,0}Y_{0,\ell}^{-1})$. Hence $V(Y_{\ell,0}Y_{0,\ell}^{-1})$ is simple as a $\mathcal{U}_q^h(sl_{n+1}^{tor})$ -module and as a $\mathcal{U}_q(sl_{n+1}^{tor})$ -module. Furthermore $v_{Y_{\ell,0}Y_{0,\ell}^{-1}}$ generates the integrable module $V(Y_{\ell,0}Y_{0,\ell}^{-1})$ and is extremal of weight ϖ_ℓ . As the $\mathcal{U}_q^h(sl_{n+1}^{tor})$ -module $V(Y_{\ell,0}Y_{0,\ell}^{-1})$ is simple, it is isomorphic to $V(\varpi_\ell)$. \square

This result suggests that the $\mathcal{U}_q(sl_{n+1}^{tor})$ -module $V(Y_{\ell,0}Y_{0,\ell}^{-1})$ can be obtained from the extremal weight module $V(\varpi_\ell)$ by an evaluation morphism. Actually this is not the case for the following reasons (which generalize arguments given in [23]): as a $\mathcal{U}_q^h(sl_{n+1}^{tor})$ -module, $V(Y_{\ell,0}Y_{0,\ell}^{-1})$ is isomorphic to $V(\varpi_\ell)$. In particular,

$$\tau_p : V(\varpi_\ell) \rightarrow V(\varpi_\ell), v_m \mapsto v_{\tau_p(m)} \text{ for all } m \in \mathcal{M}(Y_{\ell,0}Y_{0,\ell}^{-1})$$

is a $\mathcal{U}_q(\hat{sl}_{n+1})'$ -automorphism of $V(\varpi_\ell)$ (with $p = n+1, 2$ if $\ell = 1, r+1$ respectively). If $V(Y_{\ell,0}Y_{0,\ell}^{-1})$ is obtained from an evaluation morphism $\mathcal{U}_q(sl_{n+1}^{tor}) \rightarrow \mathcal{U}_q^h(sl_{n+1}^{tor})$, τ_p should induce an automorphism of $V(Y_{\ell,0}Y_{0,\ell}^{-1})$. But it does not commute with the action of the $x_{i,r}^\pm, h_{i,r}$ for $i \in I$ and $r \in \mathbb{Z} - \{0\}$. In the same way, $V(Y_{\ell,0}Y_{0,\ell}^{-1})$ can not be obtained from an evaluation morphism $\mathcal{U}_q(sl_{n+1}^{tor}) \rightarrow \mathcal{U}_q^{v,j}(sl_{n+1}^{tor})$ ($j \in I$). In fact, $V(Y_{\ell,0}Y_{0,\ell}^{-1})$ is completely reducible as a $\mathcal{U}_q^{v,j}(sl_{n+1}^{tor})$ -module and is a direct sum of fundamental modules (see (15)). But it is a simple $\mathcal{U}_q(sl_{n+1}^{tor})$ -module.

From now on, let \mathcal{M}' be a subcrystal of \mathcal{M} for $\mathcal{U}_q(sl_{n+1})$ (resp. $\mathcal{U}_q(\hat{sl}_{n+1})$) and consider the vector space V with basis (v_m) indexed by the vertices of \mathcal{M}' . We define an action of $\mathcal{U}_q(\hat{sl}_{n+1})$ (resp. $\mathcal{U}_q(sl_{n+1}^{tor})$) on V by the following formulas

$$(18) \quad \begin{aligned} x_{i,r}^+ \cdot v_m &= q^{r(p_i(m)-1)} v_{\tilde{e}_i \cdot m}, \\ x_{i,r}^- \cdot v_m &= q^{r(q_i(m)+1)} v_{\tilde{f}_i \cdot m}, \\ \phi_{i,\pm s}^\pm \cdot v_m &= \pm(q - q^{-1}) (\varphi_i(m) q^{\pm s(q_i(m)+1)} - \varepsilon_i(m) q^{\pm s(p_i(m)-1)}) v_m, \\ k_i^\pm \cdot v_m &= q^{\pm(\varphi_i(m) - \varepsilon_i(m))} v_m. \end{aligned}$$

with $r \in \mathbb{Z}$, $s > 0$ and $i \in I_0$ (resp. $i \in I$) and where $v_0 = 0$ by convention. Note that $p_i(m)$ is well defined only if $\varepsilon_i(m) > 0$ or equivalently if $\tilde{e}_i \cdot m \neq 0$ and $q_i(m)$ is well defined only if $\varphi_i(m) > 0$ or equivalently if $\tilde{f}_i \cdot m \neq 0$. Then, these expressions make sense.

Theorem 4.10. (1) Set $n \in \mathbb{N}^*$, $1 \leq \ell \leq n$. Assume that $\mathcal{M}' = \mathcal{M}(Y_{\ell,k})$ as I_0 -crystals. Then formulas (18) endow V with a structure of $\mathcal{U}_q(\hat{sl}_{n+1})$ -module isomorphic to the fundamental module $V(Y_{\ell,k})$.

(2) Assume that $n = 2r+1$ is odd and $\ell = 1$ or $\ell = r+1$. Set $\mathcal{M}' = \mathcal{M}(Y_{\ell,0}Y_{0,\ell}^{-1})$ as I -crystals. Then formulas (18) endow V with a structure of $\mathcal{U}_q(sl_{n+1}^{tor})$ -module isomorphic to the level 0 extremal fundamental loop weight module $V(Y_{\ell,0}Y_{0,\ell}^{-1})$.

Proof. The action of the horizontal quantum subalgebra and the action of the Cartan subalgebra are known on the basis $(v_m)_{m \in \mathcal{M}'}$ for the $\mathcal{U}_q(\hat{sl}_{n+1})$ -module $V(Y_{\ell,k})$ and for the $\mathcal{U}_q(sl_{n+1}^{tor})$ -module $V(Y_{\ell,0}Y_{0,\ell}^{-1})$. From (7) it is straightforward to deduce the action of the $x_{i,r}^{\pm}$ on these modules ($r \in \mathbb{Z}$). We obtain formulas (18) given only in terms of constants of the corresponding monomial crystal. \square

Remark 4.11. In [25], the algebra $\mathcal{U}_q(\hat{sl}_{\infty})$ is introduced as the quantum affinization of $\mathcal{U}_q(sl_{\infty})$. It is defined by the same generators and relations as in Definition 2.13 with the infinite Cartan matrix $C = (C_{i,j})_{i,j \in \mathbb{Z}}$ such that

$$C_{i,i} = 2, \quad C_{i,i+1} = -1, \quad C_{i+1,i} = -1, \quad C_{i,j} = 0$$

if $i - j \notin \{-1, 0, 1\}$. The representation theory of $\mathcal{U}_q(\hat{sl}_{\infty})$ is similar to the one of $\mathcal{U}_q(sl_{n+1}^{tor})$: the simple ℓ -highest weight modules are parametrized by Drinfeld polynomials. In particular, the fundamental modules can be defined and they are the inductive limit of the fundamental modules for the quantum affine algebra $\mathcal{U}_q(\hat{sl}_{n+1})$ when $n \rightarrow \infty$ (see [25, Theorem 3.8] and [25, Proposition 3.11]). So, the previous results about the fundamental modules of $\mathcal{U}_q(\hat{sl}_{n+1})$ extend directly to the case of the fundamental modules of $\mathcal{U}_q(\hat{sl}_{\infty})$.

Example 4.12. Assume that $n = 3$ and $\ell = 1$. We study the level 0 extremal fundamental loop weight module $V(Y_{1,0}Y_{0,1}^{-1})$ for $\mathcal{U}_q(sl_4^{tor})$. Let us consider the monomial crystal $\mathcal{M}(Y_{1,0}Y_{0,1}^{-1})$. It is closed and $p = 4$ in this case. Using the notations introduced above, $\mathcal{E} = \sqcup_{k \in \mathbb{Z}} \mathcal{E}_{0,k}$ and we have

$$\mathcal{E}_{0,0} = \left\{ Y_{1,0}Y_{0,1}^{-1}, Y_{2,1}Y_{1,2}^{-1}, Y_{3,2}Y_{2,3}^{-1}, Y_{0,3}Y_{3,4}^{-1} \right\}.$$

$\mathcal{E}_{0,k}$ can be deduce from $\mathcal{E}_{0,0}$ by applying τ_{4k} . In the same way, we obtain $\mathcal{E}_{j,k}$ by applying ϕ^{j+4k} to $\mathcal{E}_{0,0}$. Then the q -character of the level 0 extremal fundamental loop weight module $V(Y_{1,0}Y_{0,1}^{-1})$ is

$$\chi_q(V(Y_{1,0}Y_{0,1}^{-1})) = \sum_{k \in \mathbb{Z}} Y_{1,4k}Y_{0,1+4k}^{-1} + Y_{2,1+4k}Y_{1,2+4k}^{-1} + Y_{3,2+4k}Y_{2,3+4k}^{-1} + Y_{0,3+4k}Y_{3,4+4k}^{-1}.$$

Furthermore the action is explicitly given by the crystal $\mathcal{M}(Y_{1,0}Y_{0,1}^{-1})$ and by the formulas (18). This module is already constructed in [23].

Example 4.13. Assume that $n = 3$ and $\ell = 2$. Let us study the level 0 extremal fundamental loop weight module $V(Y_{2,0}Y_{0,2}^{-1})$ of $\mathcal{U}_q(sl_4^{tor})$. Consider the closed monomial crystal $\mathcal{M}(Y_{2,0}Y_{0,2}^{-1})$. In this case, $p = 2$ and we have

$$\mathcal{E}_{0,0} = \left\{ \begin{array}{l} Y_{2,0}Y_{0,2}^{-1}, Y_{1,1}Y_{2,2}^{-1}Y_{3,1}Y_{0,2}^{-1}, Y_{1,1}Y_{3,3}^{-1}, \\ Y_{3,1}Y_{1,3}^{-1}, Y_{1,3}^{-1}Y_{2,2}Y_{3,3}^{-1}Y_{0,2}, Y_{2,4}^{-1}Y_{0,2} \end{array} \right\}.$$

To describe all the monomials occurring in $\mathcal{M}(Y_{2,0}Y_{0,2}^{-1})$, it is sufficient to consider only the sub- $I_{\{0,1\}}$ -crystal

$$Y_{2,0}Y_{0,2}^{-1} \xrightarrow{2} Y_{1,1}Y_{2,2}^{-1}Y_{3,1}Y_{0,2}^{-1} \xrightarrow{3} Y_{1,1}Y_{3,3}^{-1}$$

and to apply the θ -twisted automorphism ϕ (Remark 3.21). The q -character of $V(Y_{2,0}Y_{0,2}^{-1})$ is

$$\begin{aligned} \chi_q(V(Y_{2,0}Y_{0,2}^{-1})) &= \sum_{k \in \mathbb{Z}} Y_{2,2k} Y_{0,2+2k}^{-1} + Y_{1,1+2k} Y_{2,2+2k}^{-1} Y_{3,1+2k} Y_{0,2+2k}^{-1} + Y_{1,3+2k}^{-1} Y_{3,1+2k} \\ &\quad + Y_{3,1+2k} Y_{1,3+2k}^{-1} + Y_{1,3+2k}^{-1} Y_{2,2+2k} Y_{3,3+2k}^{-1} Y_{0,2+2k} + Y_{2,4+2k}^{-1} Y_{0,2+2k}, \end{aligned}$$

and the action of $\mathcal{U}_q(sl_4^{tor})$ on $V(Y_{2,0}Y_{0,2}^{-1})$ is explicitly given by the crystal $\mathcal{M}(Y_{2,0}Y_{0,2}^{-1})$ and formulas (18).

Remark 4.14. As we have said, relations between monomial crystals and the set of monomials occurring in the q -character of representations are known and have combinatorial origin (see [25, 27, 40]). The above results, in particular Theorem 4.10, give one way to better understand the representation theoretical meaning of this narrow link expected in [27]. In fact, by the formulas (18) which hold for all the fundamental modules $V(Y_{\ell,k})$ of $\mathcal{U}_q(\hat{sl}_{n+1})$ and for all the level 0 extremal fundamental loop weight modules $V(Y_{\ell,0}Y_{0,\ell}^{-1})$ of $\mathcal{U}_q(sl_{n+1}^{tor})$, the knowledge of these representations is reduce to the one of the corresponding crystals $\mathcal{M}_{I_0}(Y_{\ell,k})$ and $\mathcal{M}(Y_{\ell,0}Y_{0,\ell}^{-1})$ respectively, which is totally combinatorial.

4.3. Finite-dimensional representations at roots of unity. The existence of shift automorphisms for $\mathcal{M}(Y_{\ell,0}Y_{0,\ell}^{-1})$ when $\ell = 1$ or $\ell = r + 1$ is related to finite-dimensional representations of quantum toroidal algebras at roots of unity. We explain that in this section.

So assume that $n = 2r + 1$ is odd ($r \geq 1$) and $\ell = 1$ or $\ell = r + 1$. In this case $\mathcal{M}(Y_{\ell,0}Y_{0,\ell}^{-1})$ is closed and its automorphism z_ℓ has the special form of a shift τ_{-p} with $p = n + 1$ or 2 respectively.

Set $L \geq 1$ and ϵ a primitive (pL) -root of unity. Let $\mathcal{U}_\epsilon(sl_{n+1}^{tor})$ be the algebra defined as $\mathcal{U}_q(sl_{n+1}^{tor})$ with ϵ instead of q (without divided powers).

Let $\Gamma_N : \mathbb{Z}[Y_{i,l}^\pm]_{i \in I, l \in \mathbb{Z}} \rightarrow \mathbb{Z}[Y_{i,\bar{l}}^\pm]_{i \in I, \bar{l} \in \mathbb{Z}/N\mathbb{Z}}$ be the map defined by sending the variables $Y_{i,l}^\pm$ to $Y_{i,\bar{l}}^\pm$ ($i \in I, l \in \mathbb{Z}$). For a monomial set \mathcal{S} we denote by \mathcal{S}_ϵ its image by $\Gamma_{(pL)}$.

Consider the monomial set \mathcal{E}_ϵ . By the periodicity property given by τ_{-p} , we have

$$\mathcal{E}_\epsilon = \bigsqcup_{0 \leq k \leq L-1} \tau_p^k((\mathcal{E}_{j,k})_\epsilon)$$

with $j \in I$. One checks easily that \mathcal{E}_ϵ is closed.

By specializing the representations $V(Y_{\ell,0}Y_{0,\ell}^{-1})$ at a root of unity ϵ , we obtain

Theorem 4.15. Assume that ϵ is a primitive (pL) -root of unity. There is an irreducible $\mathcal{U}_\epsilon(sl_{n+1}^{tor})$ -module $V(Y_{\ell,0}Y_{0,\ell}^{-1})_\epsilon$ of dimension $L \cdot \binom{n+1}{\ell}$ such that

$$\chi_\epsilon(V(Y_{\ell,0}Y_{0,\ell}^{-1})_\epsilon) = \sum_{m \in \mathcal{E}_\epsilon} m.$$

Furthermore there exists a basis (v_m) of $V(Y_{\ell,0}Y_{0,\ell}^{-1})_\epsilon$ indexed by \mathcal{E}_ϵ such that the action on it is given by

$$\begin{aligned} x_{i,r}^+ \cdot v_m &= \epsilon^{r(p_i(m)-1)} v_{\tilde{e}_i \cdot m}, \\ x_{i,r}^- \cdot v_m &= \epsilon^{r(q_i(m)+1)} v_{\tilde{f}_i \cdot m}, \\ \phi_{i,\pm s}^\pm \cdot v_m &= \pm(\epsilon - \epsilon^{-1}) (\varphi_i(m) \epsilon^{\pm s(q_i(m)+1)} - \varepsilon_i(m) \epsilon^{\pm s(p_i(m)-1)}) v_m, \\ k_i^\pm \cdot v_m &= \epsilon^{\pm(\varphi_i(m) - \varepsilon_i(m))} v_m. \end{aligned}$$

5. EXTREMAL LOOP WEIGHT MODULES FOR $\mathcal{U}_q(sl_{n+1}^{tor})$ WHEN THE CONSIDERED CRYSTAL IS NOT CLOSED

In the last part, we have constructed a new family of extremal loop weight modules $V(Y_{\ell,0}Y_{0,\ell}^{-1})$ of $\mathcal{U}_q(sl_{n+1}^{tor})$ when $n = 2r + 1$ is odd and $\mathcal{M}(Y_{\ell,0}Y_{0,\ell}^{-1})$ is closed (this is the case if and only if $\ell = 1$ or $\ell = r + 1$).

In this section, we still suppose that $n = 2r + 1$ is odd and we discuss the case where the considered monomial crystal \mathcal{M}' is not closed. It is not possible here to construct an integrable module whose q -character is a sum of monomials occurring in \mathcal{M}' and we have to consider a larger closed monomial crystal $\overline{\mathcal{M}'}$ containing it. One can obtain such a crystal from \mathcal{M}' by adding other monomial crystals. But its structure is more complicated than \mathcal{M}' and it is difficult for us to construct systematically the possible representation of $\mathcal{U}_q(sl_{n+1}^{tor})$ associated to $\overline{\mathcal{M}'}$.

So in this section, we propose to treat an example of such a construction. Assume in the following that $n = 3$ and consider the crystal $\mathcal{M}(Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})$ which is not closed. We determine a closed monomial crystal $\overline{\mathcal{M}}(Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})$ containing it and we construct a representation $V(Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})$ of $\mathcal{U}_q(sl_4^{tor})$ such that its q -character is the sum of the monomials occurring in $\overline{\mathcal{M}}(Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})$ with multiplicity one (Theorem 5.6). Furthermore we will see that $V(Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})$ satisfies the definition of extremal loop weight module.

In the first part, we study the crystal $\mathcal{M}(Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})$ and we determine a closed crystal $\overline{\mathcal{M}}(Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})$ containing it.

The construction of the $\mathcal{U}_q(sl_4^{tor})$ -module $V(Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})$ is done in the second part. The process is the same as in the preceding section: we consider the vector space freely generated by the vertices m of $\overline{\mathcal{M}}(Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})$ and we define an action of $\mathcal{U}_q(sl_4^{tor})$ by pasting together some finite-dimensional representations of the vertical subalgebras $\mathcal{U}_q^{v,j}(sl_4^{tor})$ ($j \in I$).

In the third part, we study the representation $V(Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})$: it is an irreducible integrable representation of $\mathcal{U}_q(sl_4^{tor})$. Furthermore $V(Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})$ is a level 0 extremal loop weight module generated by a vector v of level 0 ℓ -weight $Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1}$.

In the last part, we specialize q at roots of unity ϵ . We get finite-dimensional representations of the quantum toroidal algebra $\mathcal{U}_\epsilon(sl_4^{tor})$.

Remark 5.1. *It could be interesting to construct other level 0 extremal fundamental loop weight modules of ℓ -weight $Y_{\ell,0}Y_{0,\ell}^{-1}$ with $2 \leq \ell \leq r$ in the same way. The first*

crystal $\mathcal{M}(Y_{\ell,0}Y_{0,\ell}^{-1})$ which is not closed is obtained for $n = 5$ and $\ell = 2$. We are led to consider the following closed crystal

$$\overline{\mathcal{M}}(Y_{2,0}Y_{0,2}^{-1}) = \mathcal{M}(Y_{2,0}Y_{0,2}^{-1}) \oplus \bigoplus_{s \in \mathbb{N}^*} \mathcal{M}(Y_{1,1}Y_{1,-1+6s}Y_{0,2}^{-1}Y_{0,6s}^{-1}).$$

which contains $\mathcal{M}(Y_{\ell,0}Y_{0,\ell}^{-1})$. The maps ϕ and τ_6 are automorphisms of it and the P_{cl} -crystals $\mathcal{M}(Y_{2,0}Y_{0,2}^{-1})/\tau_6$ and $\mathcal{M}(Y_{1,1}Y_{1,-1+6s}Y_{0,2}^{-1}Y_{0,6s}^{-1})/\tau_6$ have 30 vertices and 36 vertices respectively.

The example we propose to treat in this section is simpler than the case of the level 0 extremal fundamental loop weight modules and we focus only on this situation for the sake of clarity and simplicity.

5.1. Study of the monomial $\mathcal{U}_q(\hat{sl}_4)$ -crystal $\mathcal{M}(Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})$. We refer to the Appendix for explicit descriptions of all the crystals considered in this section. Let us study the monomial crystal $\mathcal{M}(Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})$: the maps ϕ and τ_4 are automorphisms of $\mathcal{M}(Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})$, with τ_4 of weight -2δ . Furthermore straightforward computations lead to the following result.

Proposition 5.2. (1) *We have the equality of sets*

$$\mathcal{M}(Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1}) = \bigcup_{k \in \mathbb{Z}} \tau_4^k \left(\bigcup_{j=0}^3 \mathcal{M}_{I_j}(Y_{1+j,1+j}Y_{1+j,-1+j}Y_{j,2+j}^{-1}Y_{j,j}^{-1}) \right).$$

(2) *For all $j \in I$, the monomial crystal $\mathcal{M}_{I_j}(Y_{1+j,1+j}Y_{1+j,-1+j}Y_{j,2+j}^{-1}Y_{j,j}^{-1})$ is I_j - q -closed. More precisely, we have the bijection of monomial sets*

$$\Xi^j : \mathcal{M}_{I_j}(Y_{1+j,1+j}Y_{1+j,-1+j}Y_{j,2+j}^{-1}Y_{j,j}^{-1}) \longrightarrow \mathcal{M}(V(Y_{1,1+j}Y_{1,-1+j})^{(j)})$$

where $V(Y_{1,1+j}Y_{1,-1+j})^{(j)}$ is the ℓ -highest weight representation of $\mathcal{U}_q^{v,j}(sl_4^{\text{tor}})$ of ℓ -highest weight $Y_{1+j,1+j}Y_{1+j,-1+j}$.

(3) *For all $j \in I$, the I_j -crystal $\mathcal{M}_{I_j}(Y_{1+j,1+j}Y_{1+j,-1+j}Y_{j,2+j}^{-1}Y_{j,j}^{-1})$ is not q -closed: the monomial $\phi^j(Y_{1,-1}Y_{3,5}^{-1}Y_{0,0}^{-1}Y_{0,4})$ appears in this crystal, but it is not the case of $\phi^j(Y_{1,5}Y_{1,-1}Y_{0,6}^{-1}Y_{0,0}^{-1})$.*

Hence, we are led to consider the crystal $\mathcal{M}(Y_{1,1}Y_{1,-5}Y_{0,2}^{-1}Y_{0,-4}^{-1})$ which is also not closed and more generally we have to deal with all the monomial crystals $\mathcal{M}(Y_{1,1}Y_{1,-1-4s}Y_{0,2}^{-1}Y_{0,-4s}^{-1})$ with $s \in \mathbb{N}$. We set

$$\overline{\mathcal{M}}(Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1}) = \bigoplus_{s \in \mathbb{N}} \mathcal{M}(Y_{1,1}Y_{1,-1-4s}Y_{0,2}^{-1}Y_{0,-4s}^{-1}).$$

For all $(k, s) \in \mathbb{Z} \times \mathbb{N}$ and $j \in I$, denote by

- $\mathcal{M}_{j,k,s}^1$ the sub- I_j -crystal of $\mathcal{M}(Y_{1,1}Y_{1,-1-4s}Y_{0,2}^{-1}Y_{0,-4s}^{-1})$ generated by the monomial $\phi^{j+4k}(Y_{1,1}Y_{1,-1-4s}Y_{0,2}^{-1}Y_{0,-4s}^{-1})$,
- $\mathcal{M}_{j,k,s}^2$ the sub- I_j -crystal of $\mathcal{M}(Y_{1,1}Y_{1,-1-4s}Y_{0,2}^{-1}Y_{0,-4s}^{-1})$ generated by the monomial $\phi^{j+4k}(Y_{1,1}Y_{1,-3-4s}Y_{2,-4-4s}Y_{0,2}^{-1})$.

Proposition 5.3. (1) For all $s \in \mathbb{N}$ and $j \in I$, one has the equality of I_j -crystals

$$\mathcal{M}(Y_{1,1}Y_{1,-1-4s}Y_{0,2}^{-1}Y_{0,-4s}^{-1}) = \bigoplus_{k \in \mathbb{Z}} (\mathcal{M}_{j,k,s}^1 \oplus \mathcal{M}_{j,k,s}^2).$$

(2) For all $j \in I$, $k \in \mathbb{Z}$ and $s \geq 1$, the monomial crystal $\mathcal{M}_{j,k,s} = \mathcal{M}_{j,k,s}^1 \oplus \mathcal{M}_{j,k,s-1}^2$ is I_j - q -closed. More precisely, we have the bijection of monomial sets

$$\Xi^j : \mathcal{M}_{j,k,s} \longrightarrow \mathcal{M}\left(V(Y_{1,1+j+4k}Y_{1,-1+j+4k-4s})^{(j)}\right)$$

where $V(Y_{1,1+j+4k}Y_{1,-1+j+4k-4s})^{(j)}$ is the ℓ -highest weight representation of $\mathcal{U}_q^{v,j}(sl_4^{tor})$ of ℓ -highest weight $Y_{1+j,1+j+4k}Y_{1+j,-1+j+4k-4s}$.

The proof of these statements is straightforward. As a consequence of these results, we have

Corollary 5.4. The monomial crystal $\overline{\mathcal{M}}(Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})$ is closed.

Proposition 5.5. The P -crystals $\overline{\mathcal{M}}(Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})$ and $\mathcal{B}(2\varpi_1)$ are isomorphic. In particular, the monomials $Y_{1,1}Y_{1,-1-4s}Y_{0,2}^{-1}Y_{0,-4s}^{-1}$ ($s \in \mathbb{N}$) are extremal of weight $2\varpi_1$.

Proof. The monomial crystal $\mathcal{M}(Y_{1,1}^2Y_{0,2}^{-2})$ is isomorphic to the connected component of $\mathcal{B}(2\varpi_1)$ generated by $v_{2\varpi_1}$ [27, Proposition 3.1]. One checks that the map

$$\mathcal{M}(Y_{1,1}Y_{1,-1-4s}Y_{0,2}^{-1}Y_{0,-4s}^{-1}) \longrightarrow \mathcal{M}(Y_{1,1}^2Y_{0,2}^{-2})$$

which sends the monomial $Y_{1,1}Y_{1,-1-4s}Y_{0,2}^{-1}Y_{0,-4s}^{-1}$ to the extremal element $Y_{1,1}^2Y_{0,2}^{-2}$ of weight $2\varpi_1$, is an isomorphism for all $s \in \mathbb{N}$. Then the result is a direct consequence of the description of crystal bases of level 0 extremal weight modules (see [5]). \square

5.2. Construction of the $\mathcal{U}_q(sl_4^{tor})$ -module $V(Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})$. Let us give the main result of this section.

Theorem 5.6. There exists a representation of $\mathcal{U}_q(sl_4^{tor})$ whose q -character is the sum of monomials occurring in $\overline{\mathcal{M}}(Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})$ with multiplicity one. It is denoted by $V(Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})$.

The construction of $V(Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})$ is analogous to the one of $V(Y_{\ell,0}Y_{0,\ell}^{-1})$ in Theorem 4.1: we paste together the finite-dimensional representations $V(Y_{1,1+j+4k}Y_{1,-1+j+4k})^{(j)}$ and $V(Y_{1,1+j+4k}Y_{1,-1+j+4k-4s})^{(j)}$ of $\mathcal{U}_q^{v,j}(sl_4^{tor})$ with $j \in I$, $k \in \mathbb{Z}$ and $s \in \mathbb{N}^*$.

Let us begin by recalling some well-known facts about the Kirillov-Reshetikhin module $V(\Xi^0(M))$ with $M = Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1}$. It is irreducible as a $\mathcal{U}_q(sl_4)$ -module, isomorphic to $V(2\Lambda_1)$. In particular, $V(\Xi^0(M))$ is an extremal weight module of dominant weight $2\Lambda_1$ and there exists vectors $v_{\phi^i(M)}$ ($i = 0, \dots, 3$) such that v_M is an ℓ -highest

weight vector of $V(\Xi^0(M))$ and

$$\begin{aligned} (x_{i,0}^-)^{(2)} \cdot v_{\phi^{i-1}(M)} &= v_{\phi^i(M)} \text{ for } i = 1, \dots, 3, \\ (x_{i,0}^+)^{(2)} \cdot v_{\phi^i(M)} &= v_{\phi^{i-1}(M)} \text{ for } i = 1, \dots, 3, \\ (x_{i,0}^\pm)^{(2)} \cdot v_{\phi^i(M)} &= 0 \text{ in the other cases.} \end{aligned}$$

Set

$$v_{\tilde{f}_1 \cdot M} := x_{1,0}^- \cdot v_M, \quad v_{\tilde{f}_2 \tilde{f}_1 \cdot M} := x_{2,0}^- x_{1,0}^- \cdot v_M, \quad v_{\tilde{f}_3 \tilde{f}_2 \tilde{f}_1 \cdot M} := x_{3,0}^- x_{2,0}^- x_{1,0}^- \cdot v_M,$$

$$v_{\tilde{f}_2 \cdot \phi(M)} := x_{2,0}^- \cdot v_{\phi(M)}, \quad v_{\tilde{f}_3 \tilde{f}_2 \cdot \phi(M)} := x_{3,0}^- x_{2,0}^- \cdot v_{\phi(M)},$$

$$v_{\tilde{f}_3 \cdot \phi^2(M)} := x_{3,0}^- \cdot v_{\phi^2(M)}.$$

These vectors form a basis (v_m) of $V(\Xi^0(M))$, indexed by the monomials occurring in $\mathcal{M}_{I_0}(M)$. Furthermore for all $m \in \mathcal{M}_{I_0}(M)$, v_m is an ℓ -weight vector of ℓ -weight $\Xi^0(m)$.

The other finite-dimensional representations of $\mathcal{U}_q(\hat{sl}_4)$ we have to consider are $W = V(\Xi^0(M))$ with $M = Y_{1,1}Y_{1,-1-4s}Y_{0,2}^{-1}Y_{0,4s}^{-1}$ and $s \in \mathbb{N}^*$. The following two points are well-known:

- (1) W is irreducible as a $\mathcal{U}_q(\hat{sl}_4)$ -module, isomorphic to $W = V(Y_{1,1}) \otimes V(Y_{1,-1-4s})$,
- (2) W is completely reducible as a $\mathcal{U}_q(sl_4)$ -module, isomorphic to $V(2\Lambda_1) \oplus V(\Lambda_2)$.

Furthermore there exist vectors $v_{\phi^i(M)}$ ($i = 0, \dots, 3$) such that v_M is an ℓ -highest weight vector of W and

$$\begin{aligned} (x_{i,0}^-)^{(2)} \cdot v_{\phi^{i-1}(M)} &= v_{\phi^i(M)} \text{ for } i = 1, \dots, 3, \\ (x_{i,0}^+)^{(2)} \cdot v_{\phi^i(M)} &= v_{\phi^{i-1}(M)} \text{ for } i = 1, \dots, 3, \\ (x_{i,0}^\pm)^{(2)} \cdot v_{\phi^i(M)} &= 0 \text{ in the other cases.} \end{aligned}$$

To complete this family of vectors to a basis of W , the following example is used.

Example 5.7. *Let $a, b \in \mathbb{Z}$ be such that $a \neq b$ and $a \neq b \pm 2$. Consider the $\mathcal{U}_q(\hat{sl}_2)$ -module $V(Y_a Y_b)$. This module was already studied in [24]. We have*

$$\chi_q(V(Y_a Y_b)) = Y_a Y_b + Y_a Y_{b+2}^{-1} + Y_{a+2}^{-1} Y_b + Y_{a+2}^{-1} Y_{b+2}^{-1}.$$

In particular, it was shown that there exists a basis

$$\{v_{Y_a Y_b}, v_{Y_{a+2}^{-1} Y_b}, v_{Y_a Y_{b+2}^{-1}}, v_{Y_{a+2}^{-1} Y_{b+2}^{-1}}\}$$

where the action of the Drinfeld generators on it is given by

$$\begin{aligned}
x_r^+ \cdot v_{Y_a Y_b} &= 0, \\
x_r^- \cdot v_{Y_a Y_b} &= \frac{q^{b-1} - q^{a+1}}{q^b - q^a} q^{r(a+1)} v_{Y_{a+2}^{-1} Y_b} + \frac{q^{b+1} - q^{a-1}}{q^b - q^a} q^{r(b+1)} v_{Y_a Y_{b+2}^{-1}}, \\
x_r^+ \cdot v_{Y_{a+2}^{-1} Y_b} &= q^{r(a+1)} v_{Y_a Y_b}, \\
x_r^- \cdot v_{Y_{a+2}^{-1} Y_b} &= q^{r(b+1)} v_{Y_{a+2}^{-1} Y_{b+2}^{-1}}, \\
x_r^+ \cdot v_{Y_a Y_{b+2}^{-1}} &= q^{r(b+1)} v_{Y_a Y_b}, \\
x_r^- \cdot v_{Y_a Y_{b+2}^{-1}} &= q^{r(a+1)} v_{Y_{a+2}^{-1} Y_{b+2}^{-1}}, \\
x_r^+ \cdot v_{Y_{a+2}^{-1} Y_{b+2}^{-1}} &= \frac{q^{b-1} - q^{a+1}}{q^b - q^a} q^{r(b+1)} v_{Y_{a+2}^{-1} Y_b} + \frac{q^{b+1} - q^{a-1}}{q^b - q^a} q^{r(a+1)} v_{Y_a Y_{b+2}^{-1}}, \\
x_r^- \cdot v_{Y_{a+2}^{-1} Y_{b+2}^{-1}} &= 0,
\end{aligned}$$

and with v_m of ℓ -weight m for $m = Y_a Y_b, \dots, Y_{a+2}^{-1} Y_{b+2}^{-1}$. Note that the basis used in [24] is renormalized here and we have

$$(x_0^-)^{(2)} \cdot v_{Y_a Y_b} = v_{Y_{a+2}^{-1} Y_{b+2}^{-1}}, \quad (x_0^+)^{(2)} \cdot v_{Y_{a+2}^{-1} Y_{b+2}^{-1}} = v_{Y_a Y_b}.$$

As $a \neq b \pm 2$, it is well-known that $V(Y_a Y_b)$ is isomorphic to $V(Y_a) \otimes V(Y_b)$. Furthermore as a $\mathcal{U}_q(\mathfrak{sl}_2)$ -module, $V(Y_a Y_b)$ is not irreducible, but it is cyclic generated by one of vectors $v_{Y_a Y_{b+2}^{-1}}$ or $v_{Y_{a+2}^{-1} Y_b}$.

Set $M_1 = Y_{1,3}^{-1} Y_{1,-1-4s} Y_{2,2} Y_{0,-4s}^{-1}$ and $M_2 = Y_{1,1} Y_{1,1-4s}^{-1} Y_{2,-4s} Y_{0,2}^{-1}$ and let v_{M_1} and $v_{M_2} \in W$ of ℓ -weight M_1 and M_2 respectively, be such that

$$x_{1,0}^- \cdot v_M = \frac{q^{-2-4s} - q^4}{q^{-1-4s} - q^3} q^{4r} v_{M_1} + \frac{q^{-4s} - q^2}{q^{-1-4s} - q^3} q^{(-4s)r} v_{M_2}.$$

Set

$$v_{\tilde{f}_2 \cdot M_u} := x_{2,0}^- \cdot v_{M_u}, \quad v_{\tilde{f}_3 \tilde{f}_2 \cdot M_u} := x_{3,0}^- x_{2,0}^- \cdot v_{M_u}$$

with $u = 1, 2$. In the same way, one can define $v_{\phi(M_u)}, v_{\tilde{f}_3 \cdot \phi(M_u)}$ and $v_{\phi^2(M_u)}$ for $u = 1, 2$. We check that these vectors form a basis (v_m) of W , indexed by the monomials occurring in $\mathcal{M}_{0,0,s}$. Moreover v_m is an ℓ -weight vector of ℓ -weight $\Xi^0(m)$ for all m .

By twisting the action on $V(Y_{1,1} Y_{1,-1})$ and $V(Y_{1,1} Y_{1,-1-4s})$ by $\theta^{(j)}$ and τ_b for some $b \in \mathbb{C}^*$, we obtain for all $j \in I, k \in \mathbb{Z}$ and $s \in \mathbb{N}^*$

- the $\mathcal{U}_q^{v,j}(\mathfrak{sl}_4^{tor})$ -modules $V(Y_{1,1+j+4k} Y_{1,-1+j+4k})^{(j)}$ which we call modules of type KR below,
- the $\mathcal{U}_q^{v,j}(\mathfrak{sl}_4^{tor})$ -modules $V(Y_{1,1+j+4k} Y_{1,-1+j+4k-4s})^{(j)}$ which we call modules of type s -TP below. All the modules of type s -TP for various $s \in \mathbb{N}^*$ are called modules of type TP.

In particular, we get bases (v_m) of these modules indexed by the monomial crystal $\mathcal{M}_{I_j}(\phi^{j+4k}(Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1}))$ (resp. $\mathcal{M}_{j,k,s}$) with analogous properties as the previous ones. In particular, the action on a vector v_m is completely determined by the horizontal subalgebra on it and by its ℓ -weight m .

Let us begin the construction of the $\mathcal{U}_q(sl_4^{tor})$ -module $V(Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})$. Denote by \mathcal{E} the set of monomials occurring in $\overline{\mathcal{M}}(Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})$ and for all $j \in I$, $k \in \mathbb{Z}$ and $s \in \mathbb{N}^*$, $\mathcal{E}_{j,k,0}$ (resp. $\mathcal{E}_{j,k,s}$) the set of monomials corresponding to $\mathcal{M}_{j,k,0}^1$ (resp. $\mathcal{M}_{j,k,s}$). We have for all $0 \leq j \leq 3$,

$$\mathcal{E} = \bigsqcup_{k \in \mathbb{Z}} \mathcal{E}_{j,k,0} \sqcup \bigsqcup_{k \in \mathbb{Z}, s \in \mathbb{N}^*} \mathcal{E}_{j,k,s}.$$

Let

$$V(Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1}) = \bigoplus_{m \in \mathcal{E}} \mathbb{C}v_m$$

be the vector space freely generated by \mathcal{E} . For $0 \leq j \leq 3$, $k \in \mathbb{Z}$ and $s \in \mathbb{N}^*$, set $V_k^{(j)} = \bigoplus_{m \in \mathcal{E}_{j,k,0}} \mathbb{C}v_m$ (resp. $V_{k,s}^{(j)} = \bigoplus_{m \in \mathcal{E}_{j,k,s}} \mathbb{C}v_m$). Then for all $0 \leq j \leq 3$,

$$V(Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1}) = \bigoplus_{k \in \mathbb{Z}} V_k^{(j)} \oplus \bigoplus_{k \in \mathbb{Z}, s \in \mathbb{N}^*} V_{k,s}^{(j)}.$$

For all $j \in I$, we endow $V(Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})$ with a structure of a $\mathcal{U}_q^{v,j}(sl_4^{tor})$ -module as follows: for $k \in \mathbb{Z}$ and $s \in \mathbb{N}^*$, the vector space $V_k^{(j)}$ (resp. $V_{k,s}^{(j)}$) is isomorphic to $V(Y_{1,1+j+4k}Y_{1,-1+j+4k})^{(j)}$ (resp. $V(Y_{1,1+j+4k}Y_{1,-1+j+4k-4s})^{(j)}$) by identifying the corresponding bases. So $V_k^{(j)}$ (resp. $V_{k,s}^{(j)}$) is endowed with a structure of a $\mathcal{U}_q^{v,j}(sl_4^{tor})$ -module and $V(Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})$ by direct sum. We denote it by $V(Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})^{(j)}$.

Proposition 5.8. *There exists a structure of $\mathcal{U}_q(sl_4^{tor})$ -module on $V(Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})$ such that for all $j \in I$ the induced $\mathcal{U}_q^{v,j}(sl_{n+1}^{tor})$ -module is isomorphic to $V(Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})^{(j)}$. Furthermore the q -character of $V(Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})$ is*

$$\chi_q(V(Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})) = \sum_{m \in \mathcal{E}} m,$$

where \mathcal{E} is the set of monomials occurring in $\overline{\mathcal{M}}(Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})$.

Proof. The process is the same as in Theorem 4.1: to define an action of $\mathcal{U}_q(sl_4^{tor})$, we determine the action of the subalgebras $\hat{\mathcal{U}}_i$ for all $i \in I$. For that, let $j \in I$ be such that $j \neq i$. Then the action of $\hat{\mathcal{U}}_i$ on $V(Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})$ is the restriction of the action of $\mathcal{U}_q^{v,j}(sl_4^{tor})$ on $V(Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})^{(j)}$. We check that this is independent of the choice of $j \neq i$.

Let us show that this action endows $V(Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})$ with a structure of $\mathcal{U}_q(sl_4^{tor})$ -module. For that, we have to distinguish two types of monomials:

- the m such that there is no $s, s' \in \mathbb{N}$ with $s \neq s'$ and $m \in \mathcal{E}_{j,k,s} \cap \mathcal{E}_{j',k',s'}$ for some $0 \leq j, j' \leq 3$ and $k, k' \in \mathbb{Z}$. For such a monomial, the defined action on v_m comes from the same type of modules, i.e. only of modules of type KR or only on modules of type s -TP for one $s \in \mathbb{N}^*$,
- the m such that there is $s, s' \in \mathbb{N}$ with $s \neq s'$ and $m \in \mathcal{E}_{j,k,s} \cap \mathcal{E}_{j',k',s'}$ for some $0 \leq j, j' \leq 3$ and $k, k' \in \mathbb{Z}$. For such a monomial, the defined action on v_m comes from two different types of modules, i.e. of modules of type KR and of type TP or of modules of type s -TP and of type s' -TP with $s \neq s'$.

For the first ones, the same process as in Theorem 4.1 (using promotion operator) implies that the defining relations of $\mathcal{U}_q(sl_4^{tor})$ hold on it. For the other ones, this is more complicated. Such a monomial is of the form $m = \phi^{j+4k}(Y_{1,-1-4s}Y_{3,5}^{-1}Y_{0,4}Y_{0,-4s}^{-1})$ with $0 \leq j \leq 3$, $k \in \mathbb{Z}$, $s \in \mathbb{N}^*$. Promotion operator implies some relations on v_m but not all and we check directly that they are satisfied. We do not detail the calculations here. \square

5.3. Study of the $\mathcal{U}_q(sl_4^{tor})$ -module $V(Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})$.

Proposition 5.9. *The $\mathcal{U}_q(sl_4^{tor})$ -module $V(Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})$ is integrable. Moreover, it satisfies property (4) of Remark 2.4.*

Proof. For all $j \in I$, $V(Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})$ is completely reducible as a $\mathcal{U}_q^{v,j}(sl_{n+1}^{tor})$ -module and we have

$$(19) \quad V(Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})^{(j)} = \bigoplus_{s \in \mathbb{N}, k \in \mathbb{Z}} V(Y_{1,1+j+4k}Y_{1,-1+j+4k-4s})^{(j)}.$$

The representations occurring in the direct sum at the right hand side are integrable. Hence $V(Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})$ is an integrable $\mathcal{U}_q(sl_{n+1}^{tor})$ -module. Furthermore the modules of type KR (resp. of type TP) are all isomorphic as $\mathcal{U}_q(sl_4)$ -modules and satisfy property (4) of Remark 2.4. Then, it holds for $V(Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})$, i.e.

$$V(Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})_{\nu+N\alpha_i} = \{0\} \text{ for all } \nu \in P, i \in I, N \gg 0.$$

\square

Remark 5.10. *The weight spaces of $V(Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})$ are infinite-dimensional and property (3) does not hold. In fact we check that the monomials*

$$Y_{1,1+4s}Y_{1,-1-4s}Y_{0,2+4s}^{-1}Y_{0,-4s}^{-1} \in \overline{\mathcal{M}}(Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1}) \quad (s \in \mathbb{N})$$

are of weight ϖ_1 and

$$Y_{1,1+4s}Y_{1,-5-4s}Y_{0,2+4s}^{-1}Y_{0,-4-4s}^{-1} \in \overline{\mathcal{M}}(Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1}) \quad (s \in \mathbb{N})$$

are of weight $\varpi_1 + \delta$. Hence, the weight spaces of weight ϖ_1 and $\varpi_1 + \delta$ are infinite-dimensional and more generally all the weight spaces of $V(Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})$ are infinite-dimensional. Note that the ℓ -weight spaces are all of dimension one.

The main result of this section is the following.

Theorem 5.11. *The representation $V(Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})$ is a level 0 extremal loop weight module generated by the vector $v_{Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1}}$ of ℓ -weight $Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1}$.*

Proof. The third point of Definition 2.21 is a consequence of (19). For the first two points, we use the following results. \square

Lemma 5.12. *Let V be a $\mathcal{U}_q(\hat{sl}_{n+1})$ -module with basis $(v_m)_{m \in \mathcal{M}'}$ indexed by a subcrystal \mathcal{M}' of \mathcal{M} . Assume that $M \in \mathcal{M}'$ is extremal of weight λ and*

$$\text{wt}(v_m) = \text{wt}(m), \quad x_i^\pm \cdot v_m = 0 \text{ if } \pm \text{wt}(m)(h_i) \geq 0 \text{ and } (x_i^\mp)^{(\pm \text{wt}(m)(h_i))} \cdot v_m = v_{S_i(m)}$$

for all $i \in I$ and $m \in \hat{W} \cdot M$. Then v_M is an extremal vector of weight λ .

Proof. The proof is analogue as the one of Lemma 4.7. \square

Corollary 5.13. *Set $M_s = Y_{1,1}Y_{1,-1-4s}Y_{0,2}^{-1}Y_{0,-4s}^{-1}$ with $s \in \mathbb{N}$. Then v_{M_s} is an extremal vector of $V(Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})$ of weight $2\varpi_1$ for the horizontal subalgebra $\mathcal{U}_q^h(sl_4^{\text{tor}})$.*

Proof. By construction of the basis (v_m) of $V(Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})$, we have $\text{wt}(v_m) = \text{wt}(m)$ for all m . Furthermore a monomial in $\hat{W} \cdot M_s$ is of the form $\phi^{j+4k}(M_s)$ with $j \in I_0$ and $k \in \mathbb{Z}$ and we have $\text{wt}(\phi^{j+4k}(M_s)) = 2\Lambda_{j+1} - 2\Lambda_j - 2k\delta$,

$$\begin{aligned} (x_{j,0}^+)^{(2)} \cdot v_{\phi^{j+4k}(M_s)} &= v_{\phi^{j-1+4k}(M_s)} = v_{S_j(\phi^{j+4k}(M_s))}, \\ (x_{j+1,0}^-)^{(2)} \cdot v_{\phi^{j+4k}(M_s)} &= v_{\phi^{j+1+4k}(M_s)} = v_{S_{j+1}(\phi^{j+4k}(M_s))}, \\ x_i^\pm \cdot v_{\phi^{j+4k}(M_s)} &= 0 \text{ in the other cases.} \end{aligned}$$

Hence the hypotheses of the above lemma are satisfied and v_{M_s} is extremal of weight $2\varpi_1$ for the horizontal subalgebra $\mathcal{U}_q^h(sl_4^{\text{tor}})$. \square

Proposition 5.14. *The representation $V(Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})$ is cyclic as a $\mathcal{U}_q^h(sl_4^{\text{tor}})$ -module, generated by the vector $v = v_{Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1}}$.*

Proof. Consider the sub- $\mathcal{U}_q^h(sl_4^{\text{tor}})$ -module W generated by v . By construction of the basis (v_m) of $V(Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})$, $v_m \in W$ for all $m \in \mathcal{M}(Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})$. We proceed by a recursive argument: assume that for one $s \in \mathbb{N}$, we have $v_m \in W$ for all $m \in \mathcal{M}(Y_{1,1}Y_{1,-1-4t}Y_{0,2}^{-1}Y_{0,-4t}^{-1})$ with $0 \leq t \leq s$. In particular $v_{Y_{1,-1-4s}Y_{3,5}^{-1}Y_{0,4}Y_{0,-4s}^{-1}}$ is in W and by Example 5.7,

$$x_{0,0}^- \cdot v_{Y_{1,-1-4s}Y_{3,5}^{-1}Y_{0,4}Y_{0,-4s}^{-1}} = v_{Y_{1,5}Y_{1,-1-4s}Y_{0,6}^{-1}Y_{0,-4s}^{-1}} \in W.$$

In the same way $v_{\phi^k(Y_{1,-1-4s}Y_{3,5}^{-1}Y_{0,4}Y_{0,-4s}^{-1})}$ and $v_{\phi^k(Y_{1,5}Y_{1,-1-4s}Y_{0,6}^{-1}Y_{0,-4s}^{-1})}$ are in W for any $k \in \mathbb{Z}$. But all the v_m with $m \in \mathcal{M}(Y_{1,5}Y_{1,-1-4s}Y_{0,6}^{-1}Y_{0,-4s}^{-1})$ can be obtained from this family of vectors by action of $\mathcal{U}_q^h(sl_4^{\text{tor}})$: this is straightforward from Example 5.7 and the construction of the basis (v_m) . \square

Proposition 5.15. *The $\mathcal{U}_q(sl_4^{\text{tor}})$ -module $V(Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})$ is irreducible.*

Proof. Let W be a non trivial sub- $\mathcal{U}_q(sl_4^{\text{tor}})$ -module of $V(Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})$. As the ℓ -weight spaces are of dimension one, there exists $s \in \mathbb{N}$ and a monomial $m \in \mathcal{M}(Y_{1,1}Y_{1,-1-4s}Y_{0,2}^{-1}Y_{0,-4s}^{-1})$ such that $v_m \in W$.

If $s = 0$, we have already shown in the preceding proof that V is cyclic generated by v_m and $W = V$. Assume that $s \in \mathbb{N}^*$. By Example 5.7 and the construction of (v_m) , there exists $x \in \mathcal{U}_q^h(sl_4^{tor})$ such that

$$x \cdot v_m = v_{Y_{1,1}Y_{1,-1-4s}Y_{0,2}^{-1}Y_{0,-4s}^{-1}}.$$

Furthermore $\hat{\mathcal{U}}_1 \cdot v_{Y_{1,1}Y_{1,-1-4s}Y_{0,2}^{-1}Y_{0,-4s}^{-1}}$ is the simple ℓ -highest weight module of ℓ -highest weight $Y_{1,1}Y_{1,-1-4s}$ and there exists $y \in \mathcal{U}_q(sl_4^{tor})$ such that $y \cdot v_m = v_{Y_{1,1}Y_{1,-1-4s}Y_{2,-4s}Y_{0,2}^{-1}}$ with $Y_{1,1}Y_{1,-1-4s}Y_{2,-4s}Y_{0,2}^{-1} \in \mathcal{M}(Y_{1,1}Y_{1,-1-4(s-1)}Y_{0,2}^{-1}Y_{0,-4(s-1)}^{-1})$. Repeating this argument, one shows that the vector $v_{Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1}}$ is in W . By the preceding proposition we get $W = V(Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})$. \square

Proposition 5.16. *The $\mathcal{U}_q^h(sl_4^{tor})$ -module $V(Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})$ has a crystal basis isomorphic to $\mathcal{B}(2\varpi_1)$.*

Proof. Set $K = \mathbb{C}((q))$ with q an indeterminate and let A be the subring of K consisting of rational functions in K without a pole at $q = 0$. We renormalize the basis (v_m) of the $\mathbb{C}((q))$ -vector space $V(Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})$ as follows. For all $m \in \overline{\mathcal{M}}(Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})$, let w_m be the vector defined by $w_m = \frac{1}{q}v_m$ if there exists $k \in \mathbb{Z}, s \in \mathbb{N}^*$ such that $m = \phi^k(Y_{1,1}Y_{1,-1-4s}Y_{0,2}^{-1}Y_{0,-4s}^{-1})$ and $w_m = v_m$ otherwise. Set $\mathcal{B} = (w_m)_m$ and $\mathcal{L} = \bigoplus_m Aw_m$. We check directly that $(\mathcal{L}, \mathcal{B})$ is a crystal basis of the $\mathcal{U}_q^h(sl_4^{tor})$ -module $V(Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})$, isomorphic to $\mathcal{B}(2\varpi_1)$. We do not detail the calculations. \square

Remark 5.17. *All these results suggest that $V(Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})$ is isomorphic to the level 0 extremal weight module $V(2\varpi_1)$ as a $\mathcal{U}_q^h(sl_4^{tor})$ -module. One expects to prove such a result for all the level 0 extremal loop weight modules construct by the conjectural process given above.*

5.4. Finite-dimensional representations at roots of unity. Set $L \geq 1$ and let ϵ be a primitive $(4L)$ -root of unity.

Consider \mathcal{E}' the subset of \mathcal{E} defined by

$$\mathcal{E}' = \bigsqcup_{\substack{0 \leq s \leq L-2 \\ 0 \leq k \leq L-1}} \mathcal{E}_{j,k,s}^{(1)} \sqcup \mathcal{E}_{j,k,s}^{(2)}$$

Let \mathcal{E}_ϵ and \mathcal{E}'_ϵ be the images of the sets \mathcal{E} and \mathcal{E}' respectively by the map $\Gamma_{(4L)}$.

Theorem 5.18. *Assume that ϵ is a primitive $4L$ -root of unity. There exists an irreducible $\mathcal{U}_\epsilon(sl_4^{tor})$ -module V_ϵ of dimension $16L(L-1)$ such that*

$$\chi_\epsilon(V_\epsilon) = \sum_{m \in \mathcal{E}'_\epsilon} m.$$

Proof. The main difficulty is to specialize q at ϵ in the $\mathcal{U}_q^{v,j}(sl_4^{tor})$ -modules of type TP. In fact, these modules can be undefined or reducible after specialization. For better understand these phenomena, let us begin by the study of the specialized $\mathcal{U}_\epsilon(\hat{sl}_2)$ -module $V(Y_a Y_b)_\epsilon$ with $a, b \in \mathbb{Z}$. This representation is well defined if $a \notin b + 4L\mathbb{Z}$. Assume

that in the following and study $V(Y_a Y_b)_\epsilon$. If $a \notin b \pm 2 + 4L\mathbb{Z}$, this representation is irreducible. If $a \in b \pm 2 + 4L\mathbb{Z}$, it is not irreducible: in fact,

$$\mathcal{U}_\epsilon(\hat{sl}_2) \cdot v_{Y_a Y_b} = \mathbb{C}v_{Y_a Y_b} \oplus \mathbb{C}v_{Y_{a+2}^{-1} Y_b} \oplus \mathbb{C}v_{Y_{a+2}^{-1} Y_{b+2}^{-1}}$$

is an irreducible submodule of $V(Y_a Y_b)_\epsilon$.

By our study of the $\mathcal{U}_\epsilon(\hat{sl}_2)$ -module $V(Y_a Y_b)_\epsilon$, one can specialize q at ϵ in the defining relations of the action on the basis (v_m) of $V(Y_{1,1} Y_{1,-1} Y_{0,2}^{-1} Y_{0,0}^{-1})$. Moreover, one checks that

$$\mathcal{U}_\epsilon(sl_4^{tor}) \cdot v_{Y_{1,1} Y_{1,-1-4L} Y_{0,2}^{-1} Y_{0,-4L}^{-1}} = \bigoplus_{m \in \mathcal{E} - \mathcal{E}'} \mathbb{C}v_m$$

is a sub- $\mathcal{U}_\epsilon(sl_4^{tor})$ -module of $V(Y_{1,1} Y_{1,-1} Y_{0,2}^{-1} Y_{0,0}^{-1})_\epsilon$. By taking the quotient, we obtain a $\mathcal{U}_\epsilon(sl_4^{tor})$ -module

$$V_\epsilon = \bigoplus_{m \in \mathcal{E}'} \mathbb{C}v_m$$

which is irreducible: this is straightforward with the explicit formulas of the action. \square

6. FURTHERMORE POSSIBLE DEVELOPMENTS AND APPLICATIONS

In this last section, we give other promising directions to study the extremal loop weight modules for quantum toroidal algebras of general types. Moreover, we give some possible applications of the results obtained in this article. This will be done in further papers.

In our construction of level 0 extremal loop weight modules in type A , monomial realizations of crystals and promotion operators on the finite crystals have a crucial role. Let us give some results which suggest that a similar construction is possible in other types. In [27], an explicit description of monomial realizations of level 0 extremal fundamental weight crystals of quantum affine algebras is given for all the nonexceptional types. The automorphisms z_ℓ are determined in these cases. Furthermore in other types, there exists also symmetry properties for crystals arising from automorphisms of the associated Dynkin diagram (analogue of promotion operators in type A). Using that, a combinatorial process allows to obtain Kirillov-Reshetikhin crystals from crystals of finite type (see [16, 30, 42]). These symmetry properties will be useful for an similar construction of extremal loop weight modules.

As we have viewed, the level 0 extremal fundamental loop weight modules $V(Y_{\ell,0} Y_{0,\ell}^{-1})$ ($n = 2r + 1$, $r \geq 1$, $\ell = 1$ or $r + 1$) are completely reducible as $\mathcal{U}_q^{v,0}(sl_{n+1}^{tor})$ -modules: they are direct sum of fundamental modules of $\mathcal{U}_q(\hat{sl}_{n+1})$. Similar vector spaces are considered in [6] for the quantum affine algebra $\mathcal{U}_q(\hat{\mathfrak{g}})$ associated to a simple Lie algebra \mathfrak{g} over \mathbb{C} . In fact for a finite-dimensional representation V of $\mathcal{U}_q(\hat{\mathfrak{g}})'$, the vector space $V \otimes_{\mathbb{C}} \mathbb{C}[z, z^{-1}]$ is endowed with a structure of $\mathcal{U}_q(\hat{\mathfrak{g}})$ -module by using the graduation of this algebra. So, the action is very different to the one defined in this article and we do not have a way to extend this action for the quantum toroidal algebra $\mathcal{U}_q(\mathfrak{g}^{tor})$. But it would be interesting to study an analogous construction for the quantum toroidal algebra $\mathcal{U}_q(\mathfrak{g}^{tor})$. We can expect to construct other examples of extremal loop weight modules by this process.

Let us explain an other approach to construct extremal loop weight modules which could be fruitful. Let \mathfrak{g} be a Kac-Moody algebra. For an integral weight λ , one defines $\lambda_+ = \sum_{\lambda(h_i) \geq 0} \lambda(h_i) \Lambda_i$ and $\lambda_- = \lambda_+ - \lambda$. To study the extremal weight module $V(\lambda)$, Kashiwara [31] considers the tensor product $V'(\lambda) = V(\lambda_+) \otimes V(\lambda_-)$ of the highest weight module $V(\lambda_+)$ and the lowest weight module $V(\lambda_-)$. By analogy, it would be interesting to define an action of the quantum affinization $\mathcal{U}_q(\hat{\mathfrak{g}})$ on the tensor product of ℓ -highest weight module and ℓ -lowest weight module, in the spirit of [21, 22] and [12, 13, 14, 15]. This will be studied in a further paper.

An other possible direction is to study the finite-dimensional representations of Double Affine Hecke Algebras (or Cherednick algebras) at roots of unity obtained from the new finite-dimensional representations of $\mathcal{U}_\epsilon(sl_{n+1}^{tor})$ defined above, via Schur-Weyl duality [46].

In this article, we have defined promotion operators for the level 0 extremal fundamental weight crystals $\mathcal{B}(\varpi_\ell)$ in type A_n ($n \in \mathbb{N}$ odd, $1 \leq \ell \leq n$). It will interesting to discuss the existence of promotion operators for other level 0 extremal weight crystals and to discuss the uniqueness of them in the spirit of [45].

APPENDIX

In this part, we describe the monomial crystal

$$\overline{\mathcal{M}}(Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1}) = \bigoplus_{s \in \mathbb{N}} \mathcal{M}(Y_{1,1}Y_{1,-1-4s}Y_{0,2}^{-1}Y_{0,-4s}^{-1}).$$

More precisely, we represent the two connected components $\mathcal{M}(Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})$ and $\mathcal{M}(Y_{1,1}Y_{1,-5}Y_{0,2}^{-1}Y_{0,-4}^{-1})$ of $\overline{\mathcal{M}}(Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})$. Recall that all its connected components are isomorphic to each other modulo shift of weight by δ . Furthermore the map τ_4 is an automorphism of these crystals and we only give a part of them. The full crystals are obtained by applying the automorphism τ_4 . The sub- I_0 -crystals

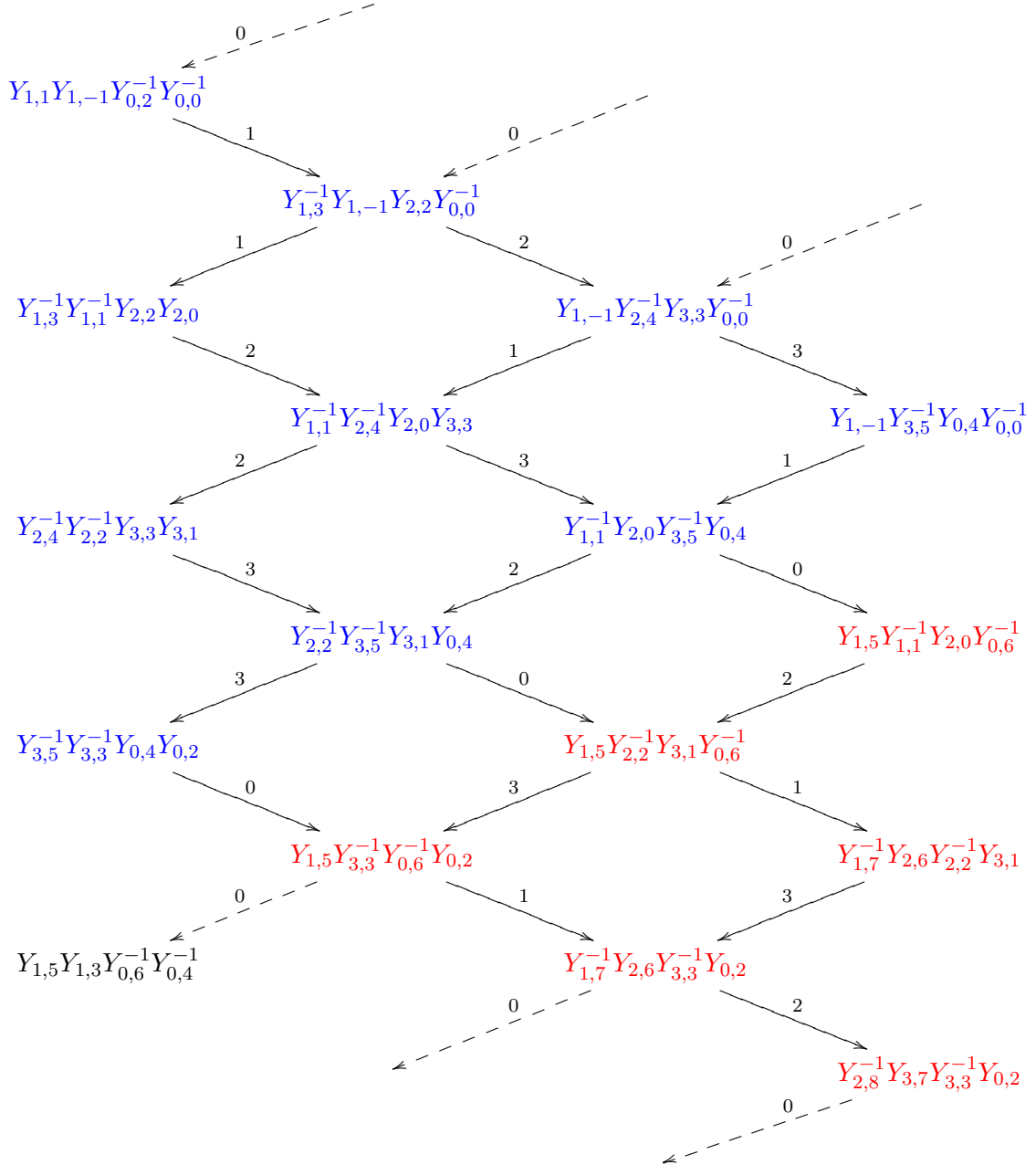
$$\mathcal{M}_{0,0,0}^1 = \mathcal{M}_{I_0}(Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})$$

and

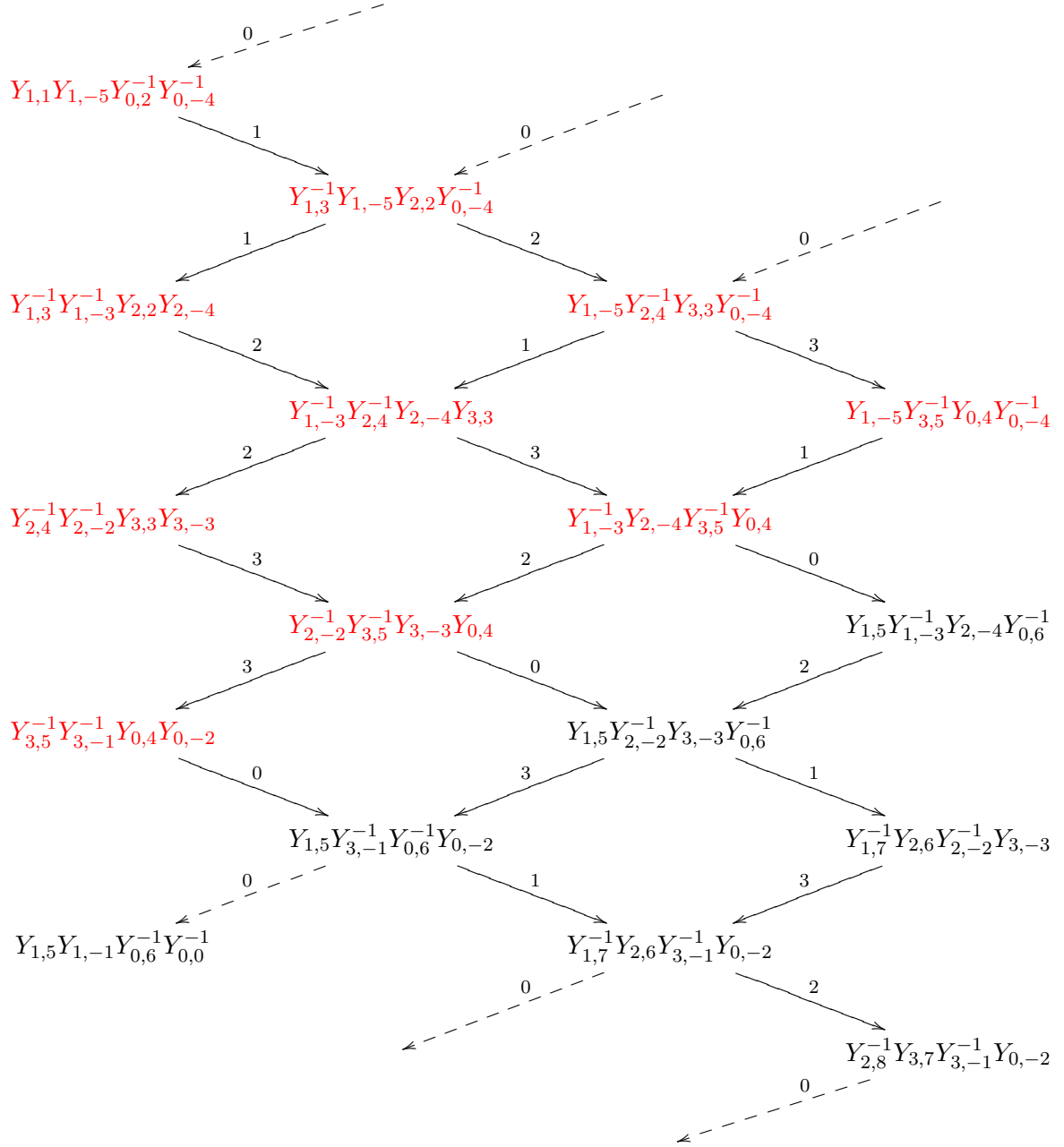
$$\mathcal{M}_{0,0,1} = \mathcal{M}_{I_0}(Y_{1,1}Y_{1,-5}Y_{0,2}^{-1}Y_{0,-4}^{-1}) \oplus \mathcal{M}_{I_0}(Y_{1,1}^{-1}Y_{1,5}Y_{2,0}Y_{0,6}^{-1})$$

are explicitly given.

Note that the θ -twisted automorphism ϕ of $\overline{\mathcal{M}}(Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})$ can be viewed as a descent of one diagonal in these crystals.



The $\mathcal{U}_q(\hat{sl}_4)$ -crystal $\mathcal{M}(Y_{1,1}Y_{1,-1}Y_{0,2}^{-1}Y_{0,0}^{-1})$.



The $\mathcal{U}_q(\hat{sl}_4)$ -crystal $\mathcal{M}(Y_{1,1}Y_{1,-5}Y_{0,2}^{-1}Y_{0,-4}^{-1})$.

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INSTITUT DE MATHÉMATIQUES DE JUSSIEU, 175, RUE DU CHEVALERET 75013 PARIS, FRANCE
 E-mail address: mansuy@math.jussieu.fr